Inverse three spectra problem for a Stieltjes string with the Neumann boundary conditions

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The notion of Stieltjes string was introduced in [1, Supplement II]. Like in [1] we suppose the string to be a thread, i. e. a string of zero density, bearing a finite number of point masses. Assume that the string consists of two parts, which are joined at one end and free to move in the direction orthogonal to the equlibrium position of the string at the other end. Starting indexing from the free ends, n_j masses $m_k^{(j)} > 0$, $k = 1, ...n_j$, are positioned on the j-th part, $j = 1, 2$, which divide the j-th part into $n_j + 1$ substrings, denoted by $l_k^{(j)} > 0$, $k = 0, ...n_j$, again starting indexing from the free end. In particular, $l_0^{(j)}$ is the distance on the j-th part between the free endpoint and $m_1^{(j)}$, $l_1^{(j)}$ for $k = 1, ..., n_j - 1$ is the distance between $m_k^{(j)}$ and $m_{k+1}^{(j)}$, and $l_{n_j}^{(j)}$ is the distance on the j-th thread between the joined end point and $m_{n_j}^{(j)}$. The tension of the thread is assumed to be equal to 1.

Considering small transverse vibrations of such string like in [2, p.55] we obtain the following spectral problem

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)} z u_k^{(j)} = 0 \quad (k = 1, 2, ..., n_j, \quad j = 1, 2), \tag{1}$$

$$u_{n_1+1}^{(1)} = u_{n_2+1}^{(2)}, \tag{2}$$

$$\sum_{j=1}^{2} \frac{u_{n_j}^{(j)} - u_{n_j+1}^{(j)}}{l_{n_j}^{(j)}} = 0, \tag{3}$$

$$u_0^{(j)} = u_1^{(j)}, \quad j = 1, 2. \tag{4}$$

where $u_k^{(j)}$ is the amplitude of vibrations of the mass $m_k^{(j)}$, z is the spectral parameter. Also we consider following two problems for the parts of the string:

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)} z u_k^{(j)} = 0 \quad (k = 1, 2, ..., n_j), \tag{5}$$

$$u_0^{(j)} = u_1^{(j)}, \tag{6}$$

$$u_{n_j+1}^{(j)} = 0. \tag{7}$$

We denote by $\{\mu_k\}_{k=1}^n$ where $n = n_1 + n_2$ the spectrum of problem (1)–(4) , and by $\{\nu_k^{(j)}\}_{k=1}^{n_j}$, $j = 1, 2$ the spectra of problems (5)–(7). Let $\{\xi_k\}_{k=1}^n = \{\nu_k^{(1)}\}_{k=1}^{n_1} \cup \{\nu_k^{(2)}\}_{k=1}^{n_2}$ be indexed such that $\xi_k \leq \xi_{k+1}$.

Location of the spectra is described by

Theorem 1. *The sequences* $\{\mu_k\}_{k=1}^n$ *and* $\{\xi_k\}_{k=1}^n$ *interlace as follows:*
1) $0=\mu_1<\xi_1\leq\mu_2\leq ...\leq\xi_n$;
2) $\mu_k=\xi_{k-1}$ *if and only if* $\mu_k=\xi_k$;
3) *the multiplicity of* ξ_k *does not exceed 2.*

The corresponding inverse problem lies in finding the masses $\{m_k^{(j)}\}_{k=1}^{n_j}$ and subintervals $\{l_k^{(j)}\}_{k=1}^{n_j}$ $(j=1,2)$ using the three spectra, i.e. spectra of problems (1)–(4), (5)–(7) with $j=1$ and (5)–(7) with $j=2$.

Theorem 2. *Let* $m_j>0 \quad (j=1,2)$ *be given together with the numbers* $\{\mu_k\}_{k=1}^n$, $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ *and* $\{\nu_k^{(2)}\}_{k=1}^{n_2-1}$ $\quad(\mu_k<\mu_{k'},\nu_k^{(j)}<\nu_{k'}^{(j)}$ *for* $k<k'$*)* $\{\xi_k\}_{k=1}^{n-1}=\{\nu_k^{(1)}\}_{k=1}^{n_1}\bigcup\{\nu_k^{(2)}\}_{k=1}^{n_2-1}$ *which satisfy the following conditions:*

$$0=\mu_1<\xi_1<\mu_2<\xi_2<...<\xi_{n-1}<\mu_n.$$

Then there exists a unique collection of sequences of positive numbers $\{\{m_k^{(j)}\}_{k=1}^{n_j}$, $(j=1,2)$, $\quad\{l_k^{(j)}\}_{k=1}^{n_j} \quad (j=1,2)\}$ *such that* $\sum\limits_{k=1}^{n_j} m_k^{(j)}=m_j$, $\{\mu_k\}_{k=1}^n$ *is the spectrum of problem (1)–(4),* $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ *is the spectrum of problem (5)–(7) with* $j=1$, $\{\nu_k^{(2)}\}_{k=1}^{n_2-1}$ *is a sequence of eigenvalues (a part of the spectrum) of problem (5)–(7) with* $j=2$.

The method of recovering $\{\{m_k^{(j)}\}_{k=1}^{n_j}, \quad (j=1,2), \quad \{l_k^{(j)}\}_{k=1}^{n_j} \quad (j=1,2)\}$ is similar to that in [3] (see also [2], p. 222).

References

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[3] O. Boyko, V. Pivovarchik. The inverse three-spectra problem for a Stieltjes string and the inverse problem with one-dimensional damping. *Inverse problems*, volume 24, no.1, 015019 13 pp, 2008.