SCATTERING IN A FORKED-SHAPED WAVEGUIDE

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Abstract. We consider wave scattering in a forked-shaped waveguide which consists of two finite and one half-infinite intervals having one common vertex. We describe the spectrum of the direct scattering problem and introduce an analogue of the Jost function. In case of the potential which is identically equal to zero on the half-infinite interval, the problem is reduced to a problem of the Regge type. For this case, using Hermite-Biehler classes, we give sharp results on the asymptotic behavior of resonances, that is, the corresponding eigenvalues of the Regge-type problem. For the inverse problem, we obtain sufficient conditions for a function to be the S-function of the scattering problem on the forked-shaped graph with zero potential on the half-infinite edge, and present an algorithm that allows to recover potentials on the finite edges from the corresponding Jost function. It is shown that the solution of the inverse problem is not unique. Some related general results in the spectral theory of operator pencils are also given.

1. Introduction

Scattering problems on graphs have been considered in many publications, see, for example, [3, 6, 7, 8, 11, 13, 14, 19, 20, 25], because of their general importance and, in particular, because of their significance in the theory of electronic micro-schemes [1, 12]. The corresponding inverse problems have been solved in [13, 20, 38, 40]. However, the problem of characterizing scattering data, i.e., the S-function, normal eigenvalues (often referred to as the energies of bound states), and normalizing constants, usually appears to be rather complicated (see [13, 20, 40]).

In the current paper we treat both direct and inverse scattering problems for the case of a simple forked-shaped graph having one half-infinite and two finite edges. The most complete results are obtained in case of the potential which is identically equal to zero on the half-infinite edge of the graph. In particular, in this case we give sufficient conditions for a set of data to be scattering data and show that these conditions are close to be necessary. Although the scattering theory for the forked-shaped graph shares many common features with the classical theory for the half-axis, it turns out that the situation considered in the current paper is essentially more complex than classical and exhibits numerous new effects. One of them is the possible presence of real eigenvalues (the bound states embedded in continuous spectrum, in terms of quantum mechanics). Another complication is the nonuniqueness of solutions of the inverse problem as described in Section 4 below. Yet another new effect is related to location of zeros of the Jost function,

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and occurs even in case of zero potential on the half-infinite edge. The latter setting corresponds to the case of a finitely supported potential in classical scattering theory where it suffices to know only a (meromorphic) scattering function to be able to recover the potential uniquely. In contrast to the classical case, for scattering on the forked-shaped graph with zero potential on the half-infinite edge, one can not claim that a meromorphic scattering function determines potentials on the finite edges uniquely even assuming that the corresponding Jost function has no real zeros (except, maybe, a simple zero at the origin); in addition, one needs to suppose that the Jost function has no pure imaginary zeros symmetric about the origin.

The following spectral problem describes one-dimensional scattering of a quantum particle when the way of propagation is a graph which consists of two finite and one half-infinite intervals (edges) having one common vertex:

$$y_j'' + (\lambda^2 - q_j(x))y_j = 0, \quad x \in [0, a], \quad j = 1, 2, \tag{1.1}$$

$$y_3'' + (\lambda^2 - q_3(x))y_3 = 0, \quad x \in [0, \infty), \tag{1.2}$$

$$y_1(\lambda, a) = y_2(\lambda, a) = y_3(\lambda, 0), \tag{1.3}$$

$$y_1'(\lambda, a) + y_2'(\lambda, a) - y_3'(\lambda, 0) = 0, \tag{1.4}$$

$$y_1(\lambda, 0) = 0, \tag{1.5}$$

$$y_2(\lambda, 0) = 0. \tag{1.6}$$

Here, λ is a complex spectral parameter, and the potentials are assumed to be real-valued and satisfy $q_j(x) \in L_2(0, a)$ for $j = 1, 2$ and $xq_3(x) \in L_1(0, \infty) \cap C[0, \infty)$.

The essential spectrum of the operator corresponding to (1.1)–(1.6) covers the positive semi-axis (and thus, since we use λ^2 as the spectral parameter in (1.1)–(1.2), the essential spectrum of problem (1.1)-(1.6) covers the real axis). In this paper we show that there may be only a finite number of normal eigenvalues of (1.1)-(1.6) lying on the imaginary axis, and a finite or infinite number of eigenvalues that belong to the essential spectrum. We construct an analogue of the S-function of classical quantum scattering theory (see [34] or [30, Chap.3]), also known as the coefficient of reflection in the theory of mechanical or electromagnetic wave propagation. Also, assuming that the potential $q_3(x)$ is identically equal to zero for $x \in [0, \infty)$, we solve the corresponding inverse problem; i.e., the problem of recovering the potentials $q_1(x)$ and $q_2(x)$ for a given $S(\lambda)$.

The paper is organized as follows. In Section 2 we first describe properties of a differential operator, A, corresponding to the boundary value problem (1.1)-(1.6) when all three potentials $q_j(x)$ are, generally, nonzero, see Theorem 2.2. Next, we introduce an analogue of the Jost function for this boundary value problem, that is, a function $E(\lambda)$ whose zeros in the lower half-plane are the normal eigenvalues of the problem, see Theorem 2.3. Also, we count in Theorem 2.4 the number of the normal eigenvalues for (1.1)-(1.6) via the number of negative eigenvalues of the Sturm-Liouville problems on each edge of the waveguide. These results use machinery from [5, 21, 22] related to the Nevanlinna, or R-functions. Finally, we introduce an analogue of the scattering, or S-function, for (1.1)-(1.6).

In Section 3 we specialize to the case when the potential $q_3(x)$ is identically equal to zero on the half-infinite edge of the waveguide. Under this condition, the zeros of the Jost function $E_0(-\lambda)$ for (1.1)–(1.6) with $q_3(x) = 0$, $x \in [0, \infty)$, constitute the spectrum of a boundary value problem of the Regge type, cf. [43]. First, we give a description of this spectrum in Theorem 3.3 using some abstract results from the theory of linear operator pencils proved in Appendix A. Next, we derive in Lemma 3.4 a representation for the Jost function $E_0(-\lambda)$ that allows us to obtain some preliminary information on asymptotics of its zeros in Lemma 3.6. Using this information and some more abstract results from Appendix A, we give in Theorem 3.7 a complete description of the geometric structure of the spectrum. In addition, we are able to prove in Theorem 3.14 that the Jost function belongs to the class of shifted symmetric generalized Hermite-Biehler functions. This fact has a number of consequences; the most notable is that the zeros of the "even" and "odd" parts of the Jost function interlace, which eventually helps to describe the asymptotic behavior of the zeros in Theorem 3.15. The information about the asymptotic behavior is, in fact, used in the sequel to setup the inverse problem for (1.1)–(1.6) with $q_3(x) = 0$, $x \in [0, \infty)$.

The inverse problem for (1.1)–(1.6) with $q_3(x) = 0$, $x \in [0, \infty)$, is solved in Section 4. First of all, we describe a way of recovering $E_0(-\lambda)$ from a given $S(\lambda)$. Next, given a function $E_0(-\lambda)$, we show how to recover potentials $q_j(x)$, $j = 1, 2$, in a way that $E_0(-\lambda)$ becomes the corresponding Jost function for (1.1)–(1.6) with $q_3(x) = 0$, $x \in [0, \infty)$. Two results of this type are proved. In a "simpler" Theorem 4.3 we are given an entire function $E_0(\lambda)$ that has a rather special representation which allows us to use the classical inverse problem results from [30, Chap.3]. In a much more involved Theorem 4.4 the given entire function $E_0(-\lambda)$ from the shifted symmetric Hermite-Biehler class is assumed to have a more general representation that resembles the representation in Lemma 3.4 used to treat the direct problem. First, we describe the asymptotic behavior of zeros of the "even" and "odd" parts of $E_0(-\lambda)$ which, again, matches the behavior seen in the direct problem. Next, we use a general fact about the Hermite-Biehler functions (see Lemma 3.11 proved in Appendix B) to show that the behavior of the zeros of the "even" and "odd" parts matches the conditions needed to apply an inverse problem result from [37], thus enabling us to recover the potentials.

Finally, in Appendix A we prove several abstract spectral results for operator pencils having some independent interest besides applications to the boundary value problem (1.1)–(1.6), and in Appendix B we collect necessary information on the Hermite-Biehler functions, give the proof of Lemma 3.11, and formulate the result from [37] used in the inverse problem part of the current paper.

2. Direct problem: general case

For an operator A on a Hilbert space, we let $D(A)$, $\rho(A)$ and $\sigma(A)$ denote its domain, resolvent set and spectrum. We refer to [16, Sec.I.2] for the definition of normal (that is, isolated Fredholm) eigenvalues, and denote by $\sigma_0(A)$ the set of normal eigenvalues of A and by $\sigma_{\mathrm{ess}}(A) = \sigma(A)\backslash\sigma_0(A)$ the essential spectrum. Some standard notions from the spectral theory of operator pencils are collected in Appendix A. At this point we recall that the spectrum of any selfadjoint operator A coincides with its approximative spectrum, see, e.g., [9, p.118], where the latter is defined as the set of $\lambda \in \mathbb{C}$ such that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $D(A)$, called *the approximate sequence* for λ, with the properties $||f_n|| = 1$ and $(\lambda I - A)f_n \to 0$ as $n \to \infty$. If the sequence $\{f_n\}_{n=1}^{\infty}$ is compact, then λ is either a normal eigenvalue, or an eigenvalue that belongs to the essential spectrum (in the latter case, in quantum mechanics, λ is called a bound state embedded into the continuous spectrum). We denote by $L_1^1(0,\infty)$ the class of functions $f(x)\in L_1(0,\infty)$ with the finite first moment $\int_0^\infty x|f(x)|dx$, and by $C([0,\infty))$ the class of continuous functions.

On the Hilbert space $L_2(0,a)\bigoplus L_2(0,a)\bigoplus L_2(0,\infty)$ of vector-valued functions $(y_j(x))_{j=1}^3$ we introduce an operator, A, related to the boundary value problem (1.1)–(1.6), acting as $A(y_i(x))_{j=1}^3=(-y_j''(x)+q_j(x)y_j(x))_{j=1}^3$ with the domain

$$D(A)=\Big\{(y_j)_{j=1}^3:\quad y_j(x)\in W_2^2(0,a), y_j(0)=0, j=1,2,\\ y_3(x)\in L_2(0,\infty), -y_3''+q_3(x)y_3(x)\in L_2(0,\infty),\\ y_1(a)=y_2(a)=y_3(0), y_1'(a)+y_2'(a)-y_3'(0)=0\Big\}, \tag{2.1}$$

where W_2^2 is the usual Sobolev space. We identify the spectrum of the operator pencil $\lambda^2 I-A$ with the spectrum of the boundary value problem (1.1)–(1.6), i.e., $\lambda\in\mathbb{C}$ is called an eigenvalue of (1.1)-(1.6) if and only if λ^2 is an eigenvalue of A.

Hypothesis 2.1. Assume that the real-valued potentials $q_j(x)$, $j=1,2,3$, satisfy conditions $q_j(x)\in L_2(0,a)$, $j=1,2$, and $q_3(x)\in L_1^1(0,\infty)\cap C[0,\infty)$.

Theorem 2.2. *Assume Hypothesis 2.1. Then the following assertions hold:*

(i) *The operator A is self-adjoint, and is bounded from below, that is, $A\geq-\beta I$, where I is the identity operator and $\beta>0$.*
(ii) $\sigma_{ess}(A)=[0,\infty)$.
(iii) *The eigenvalues of A on the essential spectrum are simple.*

Proof. First, we claim that A is symmetric. Indeed, for $Y=(y_j(x))_{j=1}^3\in D(A)$ and $Z=(z_j(x))_{j=1}^3\in D(A)$, integrating by parts, we obtain:

$$\begin{aligned}(AY,Z)=&\int_0^a y_1''\overline{z_1}dx-\int_0^a y_2''\overline{z_2}dx-\int_0^\infty y_3''\overline{z_3}dx\\ &+\int_0^a q_1y_1\overline{z_1}dx+\int_0^a q_2y_2\overline{z_2}dx+\int_0^\infty q_3y_3\overline{z_3}dx\\ &=-y_1'(a)\overline{z_1(a)}-y_2'(a)\overline{z_2(a)}+y_3'(0)\overline{z_1(0)}\\ &+\int_0^a y_1'\overline{z_1}'dx+\int_0^a y_2'\overline{z_2}'dx+\int_0^\infty y_3'\overline{z_3}'dx\\ &+\int_0^a q_1y_1\overline{z_1}dx+\int_0^a q_2y_2\overline{z_2}dx+\int_0^\infty q_3y_3\overline{z_3}dx.\end{aligned}$$

Since $Y\in D(A)$ and $Z\in D(A)$, we have $\overline{z_1}(a)=\overline{z_2}(a)=\overline{z_3}(0)$ and $y_1'(a)+y_2'(a)-y_3'(0)=0$, and therefore

$$\begin{aligned}(AY,Z)=&\int_0^a y_1'\overline{z_1}'dx+\int_0^a y_2'\overline{z_2}'dx+\int_0^\infty y_3'\overline{z_3}'dx\\ &+\int_0^a q_1y_1\overline{z_1}dx+\int_0^a q_2y_2\overline{z_2}dx+\int_0^\infty q_3y_3\overline{z_3}dx.\end{aligned}\tag{2.2}$$

Another integration by parts yields

$$\begin{aligned}(AY,Z)=&-\int_0^a y_1\overline{z_1}''dx-\int_0^a y_2\overline{z_2}''dx-\int_0^\infty y_3\overline{z_3}''dx\\ &+\int_0^a q_1y_1\overline{z_1}dx+\int_0^a q_2y_2\overline{z_2}dx+\int_0^\infty q_3y_3\overline{z_3}dx=(Y,AZ),\end{aligned}$$

proving the claim. Letting $Z = Y$ in (2.2), we obtain

$$(AY, Y) = \int_0^a |y_1'|^2 dx + \int_0^a |y_2'|^2 dx + \int_0^\infty |y_3'|^2 dx$$
$$+ \int_0^a q_1 |y_1|^2 dx + \int_0^a q_2 |y_2|^2 dx + \int_0^\infty q_3 |y_3|^2 dx. \tag{2.3}$$

Using the description of the domain of A^*, as in [42, Sec.7.5], it follows that A is self-adjoint.

The operator A is a self-adjoint extension of the operator A_0 defined by the formula $A_0 (y_i(x))_{j=1}^3 = (-y_j''(x) + q_j(x) y_j(x))_{j=1}^3$ with the domain

$$D(A_0) = \Big\{ (y_j)_{j=1}^3 : \quad y_j(x) \in W_2^2(0,a), y_j(0) = 0, j = 1,2,$$
$$y_3(x) \in L_2(0,\infty), -y_3'' + q_3(x) y_3(x) \in L_2(0,\infty),$$
$$y_1(a) = y_2(a) = y_3(0) = y_1'(a) = y_2'(a) = y_3'(0) = 0 \Big\}. \tag{2.4}$$

The operator A_0 is the direct sum of symmetric, closed, and bounded from below operators, cf., e.g., [33, Thm.V.19.5]. Therefore, A_0 is also symmetric, closed, and bounded from below (that is, $A_0 \geq -\beta_1 I$ for some $\beta_1 > 0$). Furthermore, using Theorem 16 in [33, Sec.IV], we conclude that the part of the spectrum of A located below $-\beta_1$ consists of no more than finite number of normal eigenvalues.

To prove assertion (ii), for any given $\lambda^2 \geq 0$ we construct an approximate sequence $Y_n(x)$ in $D(A)$ for λ^2 by letting $Y_n(x) = (1/3)(y_j^{(n)}(x))_{j=1}^3$, where we choose $y_3^{(n)}(x) = n^{-1/4} \exp(-n^{-1}x^2 + i\lambda x)$ and $y_j^{(n)}(x) \in W_2^2(0,a)$ so that $y_j^{(n)}(0) = 0$, $\int_0^a |y_j^{(n)}(x)|^2 dx \to 0$, $\|(y_j^{(n)})'' + (\lambda^2 - q_j(x)) y_j^{(n)}\|_{L_2(0,a)} \to 0$ as $n \to \infty$, $j = 1,2$, and, in addition, such that $y_1^{(n)}(a) = y_2^{(n)}(a) = n^{-1/4}$ and $(y_1^{(n)})'(a) + (y_2^{(n)})'(a) - i\lambda n^{-1/4} = 0$. This yields the inclusion $\sigma_{\text{ess}}(A) \supseteq [0,\infty)$. The inverse inclusion holds by Weyl's theorem since A is a relatively compact perturbation of the operator corresponding to the boundary value problem (1.1)–(1.6) with all three identically zero potentials.

To prove assertion (iii), we remark that if $\lambda^2 \geq 0$ is an eigenvalue of A, then the trivial solution $y_3(x) = 0$ is the only solution of (1.2) that belongs to $L_2(0,\infty)$. Therefore, for the corresponding eigenvector $Y(x) = (y_j(x))_{j=1}^3$ of A one has $y_3(x) = 0$, and consequently $y_1(x)$ and $y_2(x)$ satisfy the conditions $y_1(a) = y_1(0) = y_2(a) = y_2(0) = 0$ and $y_1'(a) + y_2'(a) = 0$. □

Below, we will use some special solutions of the differential equations (1.1)–(1.2). If $j = 1,2$ and $\lambda \in \mathbb{C}$ then we let $s_j(\lambda, x)$ denote the solution of (1.1) which satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$, and let $c_j(\lambda, x)$ denote the solution of (1.1) which satisfies the conditions $c_j(\lambda, 0) - 1 = c_j'(\lambda, 0) = 0$. The functions $s_j(\lambda, x)$ and $c_j(\lambda, x)$ form a fundamental system of solutions of equations (1.1), and thus for any solution $y_j(x)$ of (1.1) there exist some constants a_j, b_j such that

$$y_j(x) = a_j s_j(\lambda, x) + b_j c_j(\lambda, x), \quad x \in [0,a], \quad j = 1,2, \quad \lambda \in \mathbb{C}. \tag{2.5}$$

The Jost solutions of equation (1.2) will be denoted by $e(\lambda, x)$, $\text{Im } \lambda \geq 0$, and $e(-\lambda, x)$, $\text{Im } \lambda \leq 0$; we recall from [30, Sec.3.1] that the Jost solutions can be represented as

$$e(\lambda, x) = e^{i\lambda x} + \int_x^\infty K(x,t) e^{i\lambda t} dt, \quad \text{Im } \lambda \geq 0, \tag{2.6}$$

$$e(-\lambda, x) = e^{-i\lambda x} + \int_x^\infty K(x,t) e^{-i\lambda t} dt, \quad \text{Im } \lambda \leq 0, \tag{2.7}$$

where $K(x,t)$ is the integral kernel of a transformation operator that satisfies some well-known properties listed, e.g, in [30, Lem.3.1.1]. Moreover, the function $e(\lambda, x)$ is analytic in the open upper half-plane $\{\lambda \in \mathbb{C} : \text{Im } \lambda > 0\}$ and continuous in the closed upper half-plane $\{\lambda \in \mathbb{C} : \text{Im } \lambda \geq 0\}$, see [30, Lem.3.1.3]. If $\text{Im } \lambda = 0$ then both Jost solutions $e(\pm\lambda, x)$ are defined; moreover, if $\text{Im } \lambda = 0$ and $\lambda \neq 0$ then the functions $e(-\lambda, x)$ and $e(\lambda, x)$ form a fundamental system of solutions of equations (1.2), cf. [30, Lem.3.1.3], and thus if $y_3(x)$ is a solution of (1.2) then for some constants a_3, b_3 one has:

$$y_3(x) = a_3 e(-\lambda, x) + b_3 e(\lambda, x), \quad x \in [0, \infty), \quad \text{Im } \lambda = 0, \quad \lambda \neq 0. \tag{2.8}$$

Using notation just introduced, we define the following function $E(\lambda)$ which is analytic in the open upper half-plane $\{\lambda \in \mathbb{C} : \text{Im } \lambda > 0\}$ and continuous in the closed upper half-plane $\{\lambda \in \mathbb{C} : \text{Im } \lambda \geq 0\}$:

$$\begin{aligned} E(\lambda) =& s_1'(\lambda, a) s_2(\lambda, a) e(\lambda, 0) + s_1(\lambda, a) s_2'(\lambda, a) e(\lambda, 0) \\ &- s_1(\lambda, a) s_2(\lambda, a) e'(\lambda, 0), \quad \text{Im } \lambda \geq 0. \end{aligned} \tag{2.9}$$

Also, since $s_j(-\lambda, x) = s_j(\lambda, x)$ for $j = 1, 2$, we remark that

$$\begin{aligned} E(-\lambda) =& s_1'(\lambda, a) s_2(\lambda, a) e(-\lambda, 0) + s_1(\lambda, a) s_2'(\lambda, a) e(-\lambda, 0) \\ &- s_1(\lambda, a) s_2(\lambda, a) e'(-\lambda, 0), \quad \text{Im } \lambda \leq 0, \end{aligned} \tag{2.10}$$

and the function $E(-\lambda)$ is analytic in the open lower half-plane and continuous in the closed lower half-plane.

Theorem 2.3. *Assume Hypothesis 2.1. Then:*

(i) *The set of normal eigenvalues of problem (1.1)–(1.6)which are located in the open lower half-plane coincides with the set of zeros of the function $E(-\lambda)$ located in the open lower half-plane. In addition, these zeros belong to the imaginary axis.*

(ii) *The geometric multiplicity of any normal eigenvalue does not exceed two.*

Proof. Let us determine the normal eigenvalues of the boundary value problem (1.1)–(1.6) that belong to the open lower half plane, and correspond to solutions $(y_j(x))_{j=1}^3$ of (1.1)–(1.6) from $D(A)$. For this, see (2.5), we note that the functions $y_j(x) = a_j s_j(\lambda, x)$, $j = 1, 2$, satisfy conditions (1.5),(1.6). Next, for $\text{Im } \lambda < 0$, we need to consider two linearly independent solutions of (1.2). One of these two solutions, $e(-\lambda, x)$, is given by (2.7). The second linearly independent solution will be denoted by $\tilde{e}(-\lambda, x)$; this is the solution with the asymptotics

$$\tilde{e}(-\lambda, x) = e^{i\lambda x}(1 + o(1)) \quad \text{as} \quad x \to \infty. \tag{2.11}$$

The solution $\tilde{e}(-\lambda, x)$ exists by Theorem 7 in [33, Sec.VII.2] and grows exponentially as $x \to \infty$. Then every solution $y_3(\lambda, x)$ of (1.2) is of the form $y_3(\lambda, x) = a_3 e(-\lambda, x) + b_3 \tilde{e}(-\lambda, x)$. Since we are looking for a square summable solution $y_3(\lambda, x)$, we must have $b_3 = 0$. Substituting $y_j(x) = a_j s_j(\lambda, x)$, $j = 1, 2$ and $y_3(\lambda, x) = a_3 e(-\lambda, x)$ in the boundary conditions (1.3)–(1.4), we obtain a 3×3

system of algebraic equations for a_j, $j = 1, 2, 3$. This system has a nonzero solution if and only if λ is a normal eigenvalue of A with Im $\lambda < 0$. In turn, this happens if and only if λ is a root of the equation

$$E(-\lambda) := \det \begin{bmatrix} s_1(\lambda, a) & -s_2(\lambda, a) & 0 \\ s_1(\lambda, a) & 0 & -e(-\lambda, 0) \\ s_1'(\lambda, a) & s_2'(\lambda, a) & -e'(-\lambda, 0) \end{bmatrix} = 0. \tag{2.12}$$

This proves the first part of assertion (i). Also, the geometric multiplicity of the eigenvalue λ is equal to the dimension of the null-space of the matrix in (2.12). Since the rank of this matrix is at least one, assertion (ii) follows. Since the eigenvalues of problem (1.1)–(1.6) are square roots of the eigenvalues of the self-adjoint operator A, the eigenvalues of (1.1)–(1.6) located in the open lower half-plane must be pure imaginary, finishing the proof of assertion (i). □

Theorem 2.4. *Assume Hypothesis 2.1. Then the number n of normal eigenvalues of the boundary value problem (1.1)–(1.6) located in the open lower half-plane (counting their multiplicities) satisfies the inequalities:*

$$n_1 + n_2 + n_3 \le n \le n_1 + n_2 + n_3 + 1, \tag{2.13}$$

where n_j is the number of negative eigenvalues of the problem

$$y_j'' + (\lambda - q_j(x))y_j = 0, \quad y_j(0) = y_j(a) = 0, \quad j = 1, 2,$$

and n_3 is the number of normal negative eigenvalues of the problem

$$y_3'' + (\lambda - q_3(x))y_3 = 0, \quad y_3(0) = 0.$$

Proof. As we have seen in Theorem 2.3, the normal eigenvalues of A located in the open lower half-plane coincide with squares of the zeros of the function $E(-\lambda)$ defined in (2.10); moreover, the multiplicities of the eigenvalues coincide with the multiplicities of the zeros. We introduce the function

$$\Xi(\lambda) = -\frac{E(-\lambda)}{s_1(\lambda, a)s_2(\lambda, a)e(-\lambda, 0)}, \quad \text{Im } \lambda \le 0. \tag{2.14}$$

For β given in Theorem 2.2, we fix any $\beta_1 > \beta$ and introduce a new spectral parameter τ by the formula $\lambda = \sqrt{\tau - \beta_1}$, where we select the branch of the square root such that Im $\sqrt{\tau - \beta_1} \ge 0$ for Im $\tau \ge 0$. Using (2.10), equation (2.14) reads as follows:

$$\Xi[\tau] = -\frac{s_1'[\tau, a]}{s_1[\tau, a]} - \frac{s_2'[\tau, a]}{s_2[\tau, a]} + \frac{e'[-\tau, 0]}{e[-\tau, 0]}, \tag{2.15}$$

where we use notation $f[\pm\tau, x] = f(\pm\sqrt{\tau - \beta_1}, x)$.

We claim that $\Xi[\tau]$ is a Nevanlinna function (an R-function, in the terminology of [22]; in particular, $\Xi[\tau]$ maps the open upper half-plane into itself). To prove the claim, we remark that a sum of Nevanlinna functions is again a Nevanlinna function. That $-s_j'[\tau, a]/s_j[\tau, a]$, $j = 1, 2$, are Nevanlinna functions was proved in Lemma 2.3 of [22] for $\beta_1 = 0$. The same proof also works for $\beta_1 \ne 0$. It remains to show that $e'[\tau, 0]/e[\tau, 0]$ is a Nevanlinna function. To see this, we evaluate first its imaginary part:

(2.16)

$$\text{Im } \Big(\frac{e'[\tau, 0]}{e[\tau, 0]}\Big) = \frac{1}{2i}\Big(-\frac{\overline{e'[\tau, 0]}}{\overline{e[\tau, 0]}} + \frac{e'[\tau, 0]}{e[\tau, 0]}\Big) = \frac{\overline{e'[-\tau, 0]}e[-\tau, 0] - e'[-\tau, 0]\overline{e[-\tau, 0]}}{2i\ |e[-\tau, 0]|^2}.$$

Next, we substitute $e[-\tau,x]$ in equation (1.2), and multiply it by $\overline{e[-\tau,x]}$ to infer $e''[-\tau,x]\overline{e[-\tau,x]}+(\tau-\beta_1-q(x))\underline{e[-\tau,x]}^2=0$. Taking the imaginary part of this equation, we have $e''[\tau,x]\overline{e[\tau,x]}-\overline{e''[\tau,x]}e[\tau,x]+(\tau-\overline{\tau})|e[\tau,x]|^2=0$. Integrating, we finally have $-\overline{e'[\tau,0]}e[\tau,0]+e'[\tau,0]\overline{e[\tau,0]}=2i\text{Im}\,\tau\int_0^\infty|e[\tau,x]|^2dx$. Using arguments similar to [22, Lem.2.3], we conclude that the last term in (2.15) is indeed a Nevanlinna function, thus proving the claim.

It follows from the claim, see, e.g., [5, Thm.II.3.1], that the real poles of $\Xi[\tau]$ are simple, and there is at least one zero between any two neighboring poles. Also, it is clear that $-1/\Xi[\tau]$ is a Nevanlinna function as well. Thus, all zeros of $\Xi[\tau]$ are simple and zeros and poles of $\Xi[\tau]$ interlace. Consequently, the poles and zeros of $\Xi(\lambda)$, which lie on the interval $(-i\sqrt{\beta_1},0)$ of the imaginary axis, also interlace. Now Theorem 2.4 is proved as soon as the following assertion is verified: The smallest pole of $\Xi[\tau]$ is smaller than the smallest zero of $\Xi[\tau]$.

To prove the assertion, let τ_0 denote the smallest pole of the function $\Xi[\tau]$, that is, by (2.14), the smallest zero of the function $s_1[\tau,a]s_2[\tau,a]e[-\tau,0]$. We claim that

$$\lim_{\tau\to-\infty}\Xi[\tau]=-\infty\quad\text{while}\quad\lim_{\tau\to\tau_0,\,\tau<\tau_0}\Xi[\tau]=-\infty,\tag{2.17}$$

which implies the required assertion. The first formula in claim (2.17) follows from (2.15) and the asymptotic properties as $\tau\to-\infty$ of the functions $s_j[\tau,a]$, $j=1,2$, and $e[-\tau,0]$ and the derivatives of these functions using formulas (2.6) and (3.13), (3.15). To prove the second formula in (2.17), we consider the case when $s_1[\tau_0,a]=0$ (the cases when τ_0 is a zero of the function $s_2[\tau,a]$ or $e[-\tau,0]$ are similar). Using (2.15) and writing $s_1[\tau,a]=\dot{s}_1[\tau_0,a](\tau-\tau_0)+o(\tau-\tau_0)$ as $\tau\to\tau_0$, $\tau<\tau_0$ (here "dot" denotes $d/d\tau$), we see that the second formula in (2.17) follows from the inequality

$$\dot{s}_1[\tau_0,a]s_1'[\tau_0,a]>0.\tag{2.18}$$

Thus, it remains to prove (2.18). Applying $d/d\tau$ in the equation $s_1''[\tau,x]+(\tau-\beta_1)s_1[\tau,x]-q_1(x)s_1[\tau,x]=0$, $x\in[0,a]$, we infer $\dot{s}_1''[\tau,x]+(\tau-\beta_1)\dot{s}_1[\tau,x]+s_1[\tau,x]-q_1(x)\dot{s}_1[\tau,x]=0$. These two equations yield $s_1''[\tau,x]\dot{s}_1[\tau,x]-\dot{s}_1''[\tau,x]s_1[\tau,x]=(s_1[\tau,x])^2$. Since the left-hand side of the last formula is equal to $(s_1'[\tau,x]\dot{s}_1[\tau,x]-\dot{s}_1'[\tau,x]s_1[\tau,x])'$, integrating with respect to x from 0 to a and noting that $s_1[\tau_0,0]=\dot{s}_1[\tau_0,0]=0$ and $s_1[\tau_0,a]=0$, we have (2.18), finishing the proof of Theorem 2.4. □

Our next goal is to introduce an analogue of the scattering, or S-function for the boundary value problem (1.1)–(1.6), cf., e.g., [30, Lem.3.1.5]. The importance of this function in the classical case of scattering on the half-axis is well-known: indeed, the phase-shift, that is, the argument of the unitary S-function, is known to be a measurable quantity, see, e.g., [30, 34]. To define the S-function $S(\lambda)$ for $\text{Im}\,\lambda=0$, consider a triple $(y_j(x))_{j=1}^3$ of solutions of (1.1)–(1.2) that satisfy all four conditions (1.3)–(1.6). Formula (2.8) for $y_3(x)$ shows that, up to an independent of x multiple, $y_3(x)$ could be written as $y_3(x)=e(\lambda,x)-S(\lambda)e(-\lambda,x)$ for real $\lambda\neq0$. Here, the function $S(\lambda)$, called the *S-function*, should be chosen in a way that $y_j(x)$, $j=1,2,3$, satisfy conditions (1.3)–(1.6). Since $y_j(x)$, $j=1,2$, must satisfy (1.5),(1.6), using (2.5) we have $y_j(x)=a_js_j(\lambda,x)$, $j=1,2$. Substituting this and $y_3(x)=e(\lambda,x)-S(\lambda)e(-\lambda,x)$ in (1.3) and (1.4), we obtain a 3×3 system of equations with unknowns a_1, a_2 and $S(\lambda)$. Solving this system, we arrive at the following formula for the S-function:

$$S(\lambda) = \frac{E(\lambda)}{E(-\lambda)}, \quad \operatorname{Im} \lambda = 0. \tag{2.19}$$

The denominator $E(-\lambda)$ of this ratio is the analogue of the Jost function of classical scattering theory on the half-axis, cf. [30, Lem.3.1.5]. As we have seen in Theorem 2.2, similarly to the classical case, the zeros of $E(-\lambda)$ in the lower half-plane coincide with the normal eigenvalues of (1.1)–(1.6).

3. Direct problem: zero half-line potential

In this section we consider the case when the potential is identically equal to zero on the semi-infinite part of the waveguide, that is, throughout, we impose the following conditions on the potentials.

Hypothesis 3.1. Assume that $q_3(x) = 0$ for all $x \in [0, \infty)$ and that q_1 and q_2 are real-valued and satisfy $q_j(x) \in L_2(0, a)$, $j = 1, 2$.

Under these assumptions, assertion (ii) in Theorem 2.3 can be refined as follows.

Proposition 3.2. *If $q_3(x) = 0$, $x \in [0, \infty)$, then the operator A may have only simple normal eigenvalues.*

Proof. Under our assumption, $e(-\lambda, x) = e^{-i\lambda x}$ and therefore $e(-\lambda, 0) = 1$ and $e'(-\lambda, 0) = -i\lambda$. Using this in (2.12), we see that rank $\begin{bmatrix} s_1(\lambda,a) & -s_2(\lambda,a) & 0 \\ s_1(\lambda,a) & 0 & -1 \\ s_1'(\lambda,a) & s_2'(\lambda,a) & i\lambda \end{bmatrix} \geq 2$ since if det $\begin{bmatrix} s_1(\lambda,a) & 0 \\ s_1'(\lambda,a) & i\lambda \end{bmatrix} = 0$ then det $\begin{bmatrix} s_1(\lambda,a) & -1 \\ s_1'(\lambda,a) & i\lambda \end{bmatrix} \neq 0$. □

In the case when $q_3(x) = 0$, $x \in [0, \infty)$, the Jost function $E(-\lambda)$ defined in (2.10) will be denoted by $E_0(-\lambda)$, and could be simplified. Indeed, substituting $e(-\lambda, x) = e^{-i\lambda x}$ in (2.10) we obtain:

$$E_0(-\lambda) = s_1(\lambda, a)s_2'(\lambda, a) + s_1'(\lambda, a)s_2(\lambda, a) + i\lambda s_1(\lambda, a)s_2(\lambda, a), \ \lambda \in \mathbb{C}. \tag{3.1}$$

We remark that $E_0(\lambda)$ is symmetric, that is, one has:

$$\overline{E_0(-\overline{\lambda})} = E_0(\lambda), \quad \lambda \in \mathbb{C}. \tag{3.2}$$

The scattering function defined in (2.19) can be expressed as follows:

$$S(\lambda) = \frac{E_0(\lambda)}{E_0(-\lambda)}, \quad \lambda \in \mathbb{C}. \tag{3.3}$$

We note that if $q_3(x) = 0$ for $x \in [0, \infty)$, then $S(\lambda)$, $\lambda \in \mathbb{C}$, is a meromorphic function. The Jost function $E_0(-\lambda)$ given in (3.1) is related to the following boundary value problem of the Regge type (cf. [43]):

$$y_j'' + (\lambda^2 - q_j(x))y_j = 0, \quad x \in [0, a], \quad j = 1, 2, \tag{3.4}$$

$$y_j(\lambda, 0) = 0, \quad j = 1, 2, \tag{3.5}$$

$$y_1(\lambda, a) = y_2(\lambda, a), \tag{3.6}$$

$$y_1'(\lambda, a) + y_2'(\lambda, a) = -i\lambda y_1(\lambda, a). \tag{3.7}$$

This problem was considered in [39] for the case when all eigenvalues are located in the upper half-plane. The set of zeros of $E_0(-\lambda)$, located in the open lower half-plane, coincides with the part of the spectrum in the open lower half-plane of the Regge-type problem (3.4)–(3.7). Indeed, to see this, let us notice that, because of

$q_3(x) = 0$, $x \in [0, \infty)$, an eigenvalue λ of (3.4)–(3.7) with $\operatorname{Im} \lambda < 0$ should have an eigenvector $(y_j(\lambda, x))_{j=1}^3$ with $y_3(\lambda, x) = Ce^{-i\lambda x}$. Substituting this into (1.3) and (1.4) we obtain equalities $y_1(\lambda, a) = y_2(\lambda, a) = C$ and $y_1'(\lambda, a) + y_2'(\lambda, a) = -i\lambda C$, which is equivalent to (3.6), (3.7).

On the Hilbert space $H = L_2(0, a) \oplus \mathbb{C} \oplus L_2(0, a)$ we introduce operators A_0 and A_{00}, acting by the formulae

$$A_0 \begin{pmatrix} y_1(x) \\ y_1(a) \\ y_2(x) \end{pmatrix} = A_{00} \begin{pmatrix} y_1(x) \\ y_1(a) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} -y_1''(x) + q_1(x) y_1(x) \\ y_1'(a) + y_2'(a) \\ -y_2''(x) + q_2(x) y_2(x) \end{pmatrix}, \tag{3.8}$$

with the domains given as follows (we use $\top$ to denote transposed vectors):

$$D(A_0) = \Big\{ (y_1(x), y_1(a), y_2(x))^\top \in H : y_j(x) \in W_2^2(0, a), \quad j = 1, 2, \\ y_1(a) = y_2(a), y_1(0) = y_2(0) = 0 \Big\}, \tag{3.9}$$

$$D(A_{00}) = \Big\{ (y_1(x), y_1(a), y_2(x))^\top \in H : y_j(x) \in W_2^2(0, a), \quad j = 1, 2, \\ y_1(a) = y_2(a) = y_1'(a) = y_2'(a) = 0, y_1(0) = y_2(0) = 0 \Big\}. \tag{3.10}$$

By [33, Chap.5], A_{00} is a closed symmetric minimal and bounded from below (cf. [33, Thm.V.19.5]) operator with the defect indices $(4, 4)$, while A_0 is a self-adjoint extension of A_{00}. Hence, the spectrum of A_0 consists only of normal eigenvalues, and has no more than finitely many negative eigenvalues. Moreover, there exists a positive constant β such that $A_0 + \beta I > 0$ and the inverse operator $(A_0 + \beta I)^{-1}$ is compact. Let K and P denote the following operators:

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Clearly, $P \geq 0$, $K \geq 0$ and $P + K = I$. Let us consider the following quadratic operator pencil,

$$L(\lambda) = \lambda^2 P - i\lambda K - A_0, \tag{3.11}$$

with the domain $D(L(\lambda)) = D(A_0)$ which is independent of λ and dense in H. We collected in Appendix A several definitions and proved some abstract results from the spectral theory of operator pencils needed in the sequel. We remark that the operator pencil (3.11) satisfies Hypothesis A.1 imposed in Appendix A. Also, we identify the spectrum of the boundary value problem in (3.4)–(3.7) with the spectrum of the operator pencil $L(\lambda)$ introduced in (3.11).

Theorem 3.3. *Assume Hypothesis 3.1. Then:*

(i) *The spectrum of (3.4)-(3.7) consists only of normal eigenvalues.*
(ii) *The geometric multiplicity of each of the eigenvalues is one.*
(iii) *The spectrum of (3.4)-(3.7) is symmetric with respect to the imaginary axis, and symmetrically located eigenvalues have equal algebraic multiplicities.*
(iv) *The part of the spectrum of (3.4)-(3.7) in the open lower half-plane lies on the imaginary axis.*
(v) *The spectrum of (3.4)-(3.7) in the open lower half-plane is semi-simple.*

(vi) *The total algebraic multiplicity of the spectrum of (3.4)-(3.7) in the open lower half-plane coincides with that of the following Dirichlet problem:*

$$y'' + (\lambda^2 - q(x))y = 0, \quad x \in [0, 2a], \quad y(\lambda, 0) = y(\lambda, 2a) = 0, \tag{3.12}$$

where $q(x) = q_1(x)$ *if* $x \in [0, a)$ *and* $q(x) = q_2(2a - x)$ *if* $x \in [a, 2a]$.

Proof. To prove assertion (i), it is enough to apply Theorem 4.2 of [15, Chap.XI] to the operator pencil

$$\begin{aligned} - (A_0 + \beta I)^{-\frac{1}{2}} L(\lambda)(A_0 + \beta I)^{-\frac{1}{2}} &= I - \beta(A_0 + \beta I)^{-1} \\ &+ i(A_0 + \beta I)^{-\frac{1}{2}} K(A_0 + \beta I)^{-\frac{1}{2}} - \lambda^2 (A_0 + \beta I)^{-\frac{1}{2}} P(A_0 + \beta I)^{-\frac{1}{2}} \end{aligned}$$

which has the same spectrum as $L(\lambda)$. Assertion (ii) follows since there exists only one linearly independent solution of (3.12). Assertion (iii) holds due to the symmetry of the problem (recall that the functions $q_j(x)$ are real-valued). Assertion (iv) follows from Lemma A.3. Assertion (v) is a particular case of Lemma A.4. Since the square of the spectrum of problem (3.12) is, in fact, equal to the spectrum of the operator pencil $\lambda P - A_0$, assertion (vi) follows from Corollary A.9. □

For $j = 1, 2$, we will use the following integral representations (see [30, Sec.1.2], in particular, formula (1.2.11) therein):

$$s_j(\lambda, x) = \lambda^{-1} \sin \lambda x + \int_0^x K_j(x, t) \lambda^{-1} \sin \lambda t dt \tag{3.13}$$

$$= \lambda^{-1} \sin \lambda x - K_j(x, x)\lambda^{-2} \cos \lambda x + \int_0^x (K_j)_t(x, t)\lambda^{-2} \cos \lambda t dt, \tag{3.14}$$

$$s_j'(\lambda, x) = \cos \lambda x + K_j(x, x)\lambda^{-1} \sin \lambda x + \int_0^x (K_j)_x(x, t)\lambda^{-1} \sin \lambda t dt, \tag{3.15}$$

where we let $K_j(x, t) = 0$ for $|t| > |x|$, and, otherwise,

$$K_j(x, t) = R_j(x, t) - R_j(x, -t), \tag{3.16}$$

and $R_j(x, t)$ is the unique solution of the following integral equation:

$$R_j(x, t) = \frac{1}{2} \int_0^{\frac{x+t}{2}} q_j(\alpha) d\alpha + \int_0^{\frac{x+t}{2}} d\alpha \int_0^{\frac{x-t}{2}} q_j(\alpha + \beta) R_j(\alpha + \beta, \alpha - \beta) d\beta. \tag{3.17}$$

If $q_j(x) \in W_2^1(0, a)$, $j = 1, 2$, then, integrating by parts in (3.14) and (3.15),

$$\begin{aligned} s_j(\lambda, x) &= \lambda^{-1} \sin \lambda x - K_j(x, x)\lambda^{-2} \cos \lambda x \\ &+ (K_j)_t(x, x)\lambda^{-3} \sin \lambda x - \int_0^x (K_j)_{tt}(x, t)\lambda^{-3} \sin \lambda t dt, \end{aligned} \tag{3.18}$$

$$\begin{aligned} s_j'(\lambda, x) &= \cos \lambda x + K_j(x, x)\lambda^{-1} \sin \lambda x - (K_j)_x(x, x)\lambda^{-2} \cos \lambda x \\ &+ \int_0^x (K_j)_{xt}(x, t)\lambda^{-2} \cos \lambda t dt. \end{aligned} \tag{3.19}$$

Let $\mathcal{L}^\sigma$ denote the class of entire functions of exponential type no greater than σ which belong to $L_2(-\infty, \infty)$ for real values of the argument.

Lemma 3.4. *If Hypothesis 3.1 holds,* $E_0(\lambda)$ *is defined in (3.1), and* F_j *are defined by* $F_j = (1/2) \int_0^a q_j(x)\, dx$, $j = 1, 2$, *then:*

(i) *The following representation holds:*

$$\begin{aligned} E_0(\lambda) =&\lambda^{-1}\sin 2\lambda a-(F_1+F_2)\lambda^{-2}\cos 2\lambda a+\psi_0(\lambda)\lambda^{-2}-i(\lambda^{-1}\sin^2\lambda a\\ &-(F_1+F_2)/2\cdot\lambda^{-2}\sin 2\lambda a+\psi_1(\lambda)\lambda^{-2}),\quad \textit{where}\quad \psi_k(\lambda)\in\mathcal{L}^{2a},\ k=0,1. \end{aligned}\tag{3.20}$$

(ii) *If* $q_j(x)\in W_2^1(0,a)$, $j=1,2$, *then*

$$\begin{aligned} E_0(\lambda)=&\ \lambda^{-1}\sin 2\lambda a-(F_1+F_2)\lambda^{-2}\cos 2\lambda a\\ &+\Big((K_1)_t(a,a)+(K_2)_t(a,a)-(K_1)_x(a,a)\\ &\quad-(K_2)_x(a,a)-2F_1F_2\Big)\lambda^{-3}\sin 2\lambda a+\psi_0(\lambda)\lambda^{-3}\\ &-i\Big(\lambda^{-1}\sin^2\lambda a-(F_1+F_2)/2\cdot\lambda^{-2}\sin 2\lambda a\\ &\quad+\Big((K_1)_t(a,a)+(K_2)_t(a,a)\Big)\lambda^{-3}\sin^2\lambda a+F_1F_2\lambda^{-3}\cos^2\lambda a\Big)\\ &\quad+\psi_1(\lambda)\lambda^{-3}\sin\lambda a+\psi_2(\lambda)\lambda^{-4},\quad \textit{where}\quad \psi_k(\lambda)\in\mathcal{L}^{2a},\ k=0,1,2. \end{aligned}\tag{3.21}$$

Proof. We obtain assertion (i) by substituting (3.13)–(3.16) with $x=a$ in (3.1), and assertion (ii) by substituting (3.18) and (3.19) in (3.1) and taking into account that $\int_0^a f(t)\sin\lambda t dt\in\mathcal{L}^a$ whenever $f\in L_2(0,a)$ by the Paley-Wiener theorem. □

In what follows we will use notation

$$\widehat{S}(\lambda)=\frac{E_0(\lambda)}{E_0(-\lambda)}\cdot\frac{2\cos\lambda a+i\sin\lambda a}{2\cos\lambda a-i\sin\lambda a}.\tag{3.22}$$

Corollary 3.5. *Assume Hypothesis 3.1 and* $\lambda\in\mathbb{R}$. *Then:*

(i) $|E_0(-\lambda)-\lambda^{-1}(\sin 2\lambda a+i\sin^2\lambda a)|=O(|\lambda|^{-2})$ *as* $\lambda\to\pm\infty$.

(ii) $|\widehat{S}(\lambda)-1|=O(|\lambda|^{-1})$ *as* $\lambda\to\pm\infty$.

Proof. Assertions (i) and (ii) follow directly from (3.20). □

Next, we will describe the spectrum of (3.4)–(3.7), that is, zeros of $E_0(-\lambda)$.

Lemma 3.6. *Assume Hypothesis 3.1. Then:*

(i) *The set* $\Lambda=\{\lambda_k\}_{k=-\infty,k\neq0}^{\infty}$ *of zeros of the function* $E_0(-\lambda)$ *is contained in the horizontal strip* $|Im\ \lambda|\le M$ *for some* $M>0$.

(ii) *The zeros of the function* $E_0(-\lambda)$ *satisfy* $\lambda_{-k}=-\overline{\lambda}_k$ *for all not pure imaginary* λ_k, *and the sequence* Λ *can be split into two subsequences,* $\Lambda=\{\lambda_{2k-1}\}_{k=-\infty}^{\infty}\cup\{\lambda_{2k}\}_{k=-\infty,k\neq0}^{\infty}$, *with the following asymptotic behavior:*

$$\lambda_{2k-1}=\lambda_{2k-1}^{(0)}+o(1),\quad \lambda_{2k}=\lambda_{2k}^{(0)}+o(1)\quad \textit{as}\quad |k|\to\infty,\quad \textit{where}\tag{3.23}$$

$$\lambda_{2k-1}^{(0)}=(\pi(2k-1)+i\ln 3)(2a)^{-1},\lambda_{2k}^{(0)}=\pi k/a,\lambda_{-k}^{(0)}=-\overline{\lambda_k^{(0)}},k=1,2,\dots.\tag{3.24}$$

Proof. Suppose there exists a subsequence $\{\lambda_{k_m}\}$ of the sequence $\{\lambda_k\}$ such that Im $\lambda_{k_m}\to\infty$ as $m\to\infty$. Then (3.20) implies

$$E_0(-\lambda_{k_m})+(4i\lambda_{k_m})^{-1}\exp(-2i\lambda_{k_m}a)=o\Big(|\lambda_{k_m}|^{-1}\exp(2|\text{Im }\lambda_{k_m}a|)\Big),m\to\infty,$$

contradicting the identity $E_0(-\lambda_{k_m})=0$ and proving that the set $\{\text{Im }\lambda_k\}$ is bounded from above. Similarly, it is bounded from below, and thus assertion (i) holds. Turning to the proof of assertion (ii), we temporarily introduce the function $E_{00}(-\lambda)=\lambda^{-1}\sin 2\lambda a+i\sin^2\lambda a$ whose zeros form the sequence $\{\lambda_k^{(0)}\}_{k=-\infty,k\neq0}^{\infty}$

given in (3.24). Comparing $E_{00}(-\lambda)$ and (3.20), we conclude that there exist constants $C>0$ and $\varepsilon>0$ such that the inequality $|E_0(-\lambda)-E_{00}(-\lambda)|<C|\lambda|^{-2}$ holds for all $\lambda\in\Pi$, where $\Pi=\{\lambda:|\mathrm{Im}\,\lambda|\le M+\varepsilon, |\lambda|\ge\varepsilon\}$. For every $r\in(0,\varepsilon)$ one can find a $d>0$ such that $\left|\sin 2\lambda a+i\sin^2\lambda a\right|>d$ for all $\lambda\in\Pi\backslash\cup_k C_k$, where C_k are the disks of radii r centered at $\lambda_k^{(0)}$. Consequently, we have the inequalities $|E_{00}(-\lambda)|>d/|\lambda|>C/|\lambda|^2>|E_0(-\lambda)-E_{00}(-\lambda)|$ for all $\lambda\in\{\lambda:\lambda\in\Pi\backslash\cup_k C_k,\ |\lambda|>C/d\}$. Since r can be chosen arbitrary small, we can apply Rouché Theorem to conclude that $\lambda_k-\lambda_k^{(0)}=o(1)$ as $|k|\to\infty$. □

In fact, the spectrum of the boundary value problem (3.4)–(3.7) admits even more detailed description given next.

Theorem 3.7. *Assume Hypothesis 3.1. The spectrum of problem (3.4)–(3.7) is equal to* $\Lambda^{(1)}\cup\Lambda^{(2)}$*, where the sequences*

$$\Lambda^{(1)}=\{\lambda_k^{(1)}:k=\pm1,\pm2,\dots\},\ \Lambda^{(2)}=\{\lambda_l^{(2)}:l=\pm1,\pm2,\cdots\pm p\},\ p\le\infty,$$

satisfy the following properties:

1. *All but finitely many elements of the sequence* $\Lambda^{(1)}$ *belong to the open upper half-plane; the number of the elements of* $\Lambda^{(1)}$ *that belong to the closed lower half-plane will be denoted by* κ_1*;*
2. *All* κ_1 *elements of the sequence* $\Lambda^{(1)}$ *that belong to the closed lower half-plane are purely imaginary and occur only once; if* $\kappa_1\ge1$ *then we denote these elements by* $\lambda_{-j}^{(1)}=-i|\lambda_{-j}^{(1)}|$, $j=1,\dots,\kappa_1$*, and enumerate them such that* $|\lambda_{-j}^{(1)}|<|\lambda_{-(j+1)}^{(1)}|$ *for* $j=1,\dots,\kappa_1-1$.
3. *If* $\kappa_1\ge1$ *then the complex conjugates,* $i|\lambda_{-j}^{(1)}|$, $j=1,\dots,\kappa_1$*, of the elements listed in item (2) do not belong to the sequence* $\Lambda^{(1)}$ *(with a possible exception of* $\lambda_{-1}^{(1)}=0$*).*
4. *If* $\kappa_1\ge2$ *then the interval* $(i|\lambda_{-j}^{(1)}|,i|\lambda_{-(j+1)}^{(1)}|)$, $j=1,\dots,\kappa_1-1$*, of the imaginary axis contains an odd number of elements of the sequence* $\Lambda^{(1)}$.
5. *If* $|\lambda_{-1}^{(1)}|>0$ *then the interval* $(0,i|\lambda_{-1}^{(1)}|)$ *of the imaginary axis either contains no elements of the sequence* $\Lambda^{(1)}$*, or contains an even number of elements of this sequence.*
6. *If* $\kappa_1\ge1$ *then the interval* $(i|\lambda_{-\kappa_1}^{(1)}|,i\infty)$ *of the imaginary axis contains an odd number of elements of the sequence* $\Lambda^{(1)}$.
7. *If* $\kappa_1=0$ *then the sequence* $\Lambda^{(1)}$ *has an even number of elements with positive imaginary parts.*
8. *The numbers* $(\lambda_l^{(2)})^2$ *are real for all* $l=\pm1,\pm2,\cdots\pm p$, $p\le\infty$.
9. *The numbers* $(\lambda_l^{(2)})^2$ *can be enumerated such that*

$$(\lambda_1^{(2)})^2<(\lambda_2^{(2)})^2<...<(\lambda_{\kappa_2}^{(2)})^2<0\le(\lambda_{\kappa_2+1}^{(2)})^2<(\lambda_{\kappa_2+2}^{(2)})^2<...<(\lambda_p^{(2)})^2,$$

where $\lambda_{-l}^{(2)}=-\lambda_l^{(2)}$, $l=\pm1,\pm2,\cdots\pm p$, $p\le\infty$.

Proof. Let us define the function $E_0(-\lambda,\eta)$, $\eta\in[0,1]$, by

$$E_0(-\lambda,\eta)=s_1(\lambda,a)s_2'(\lambda,a)+s_1'(\lambda,a)s_2(\lambda,a)+i\eta\lambda s_1(\lambda,a)s_2(\lambda,a),$$

and analyze the behavior of its zeros when the parameter η changes from 0 to 1. When $\eta=0$ then all zeros of $E_0(-\lambda,0)$ are real or pure imaginary because they are the eigenvalues of problem (3.12). Among them we select those for which either $E_0(-\lambda,0)=s_1(\lambda,a)=0$ or $E_0(-\lambda,0)=s_2(\lambda,a)=0$, denote them by $\lambda_l^{(2)}$, and put them in the sequence $\Lambda^{(2)}$. All other zeros will form the sequence $\Lambda^{(1)}$; they will be denoted by $\lambda_k^{(1)}$. The zeros of $E_0(-\lambda,\eta)$ from the sequence $\Lambda^{(2)}$ do not move when η changes from 0 to 1. If $\lambda_k^{(1)}=\lambda_k^{(1)}(0)$ belongs to the sequence $\Lambda^{(1)}$ then the zeros $\lambda_k^{(1)}(\eta)$ have the following property: For all $\eta>0$, if $-i\tau\in\Lambda^{(1)}$ for some $\tau>0$ is one of these zeros, then $i\tau$ is not a zero from the sequence $\Lambda^{(1)}$. To finish the proof, we use the symmetry of the problem, and Lemmas A.3, A.4, A.8. □

Corollary 3.8. *Assume Hypothesis 3.1. Then:*

(1) *The S-function (3.3) is a meromorphic function in $\mathbb{C}$ which is continuous on $\mathbb{R}$ and has no real zeros.*
(2) *The set of poles of $S(\lambda)$ is the set of zeros of $E_0(-\lambda)$ excluding real zeros and imaginary zeros symmetric about the origin; this set satisfies properties (1)-(7) in Theorem 3.7 for $\Lambda^{(1)}$.*

Proof. Represent the numerator and the denominator of the fraction $S(\lambda)$ defined in (3.3) as products of linear terms corresponding to their zeros. The terms that correspond to the real and pure imaginary symmetric about the origin zeros of $E_0(-\lambda)$ in the denominator of $S(\lambda)$ will cancel the terms in the numerator of $S(\lambda)$ that correspond to the zeros of $E_0(\lambda)$. □

Next, we will involve in the ongoing discussion the class of Hermite-Biehler functions and its modifications, cf. [5, 21, 22]. As we will see below, the Jost function for the boundary value problem (1.1)–(1.6) with $q_3(x)=0$, $x\in[0,\infty)$, belongs to an appropriately modified class of the Hermite-Biehler functions. The main advantage of this is that we can establish important interlacing properties of zeros of the "even" and "odd" parts of the Jost function. The importance of these properties becomes especially transparent in the next section where they are used to obtain results on the inverse problem. We recall definitions from [26, p.307] and [26, p.313].

Definition 3.9. An entire function $\omega(\lambda)$ with no zeros in the closed lower half-plane and satisfying the condition $|\omega(\lambda)/\bar\omega(\lambda)|<1$ for all Im $\lambda>0$ is called a *Hermite-Biehler function* (for short, an HB-function, or a function of the HB-class). An entire function $\omega(\lambda)$ with no zeros in the open lower half-plane and satisfying the condition $|\omega(\lambda)/\bar\omega(\lambda)|\le 1$ for all Im $\lambda>0$ is called a *generalized Hermite-Biehler function* (for short, $\overline{HB}$-function).

Here, $\bar\omega(\lambda)$ denotes the entire function obtained from $\omega(\lambda)$ by replacing the coefficients in its Tailor series by their complex-conjugates, i.e. $\overline{\omega}(\lambda)=\overline{\omega(\overline{\lambda})}$.

Definition 3.10. A Hermite-Biehler function $\omega(\lambda)$ (respectively, a generalized Hermite-Biehler function) is called symmetric or a function of the SHB-class (respectively, $\overline{SHB}$-class) if $\omega(-\overline{\lambda})=\overline{\omega(\lambda)}$.

For a symmetric function $\omega(\lambda)$ one has the following representations:

$$\omega(\lambda)=P(\lambda)+iQ(\lambda)=P(\lambda)+i\lambda\hat Q(\lambda)=\tilde P(\lambda^2)+i\lambda\tilde Q(\lambda^2), \tag{3.25}$$

where $P(\lambda)$ and $\hat Q(\lambda)$ are real (that is, having real values for real λ's) and even functions. Here, we introduce the functions $\tilde P$ and $\tilde Q$ as follows:

$$\tilde P(\lambda^2)=P(\lambda),\quad \tilde Q(\lambda^2)=\hat Q(\lambda). \tag{3.26}$$

The proof of the following lemma can be found in Appendix B.

Lemma 3.11. *If an entire function $\omega(\lambda) = P(\lambda) + iQ(\lambda)$ of form (3.25) belongs to the class SHB (respectively, $\overline{SHB}$), then the entire function $\tilde{P}(\lambda) + i\tilde{Q}(\lambda)$ with $\tilde{P}$ and $\tilde{Q}$ given in (3.26), belongs to the class HB (respectively, $\overline{HB}$).*

Definition 3.12. Assume that the function $\omega(\lambda) = \tilde{P}(\lambda^2) + i\lambda\tilde{Q}(\lambda^2)$ belongs to the SHB-class. Then the function $\omega_c(\lambda) = \tilde{P}(\lambda^2+c) + i\lambda\tilde{Q}(\lambda^2+c)$ with some $c > 0$ is called a *shifted symmetric Hermite-Biehler function* (for short, SHB_c-function).

Definition 3.13. (see [27]) An entire function $\omega(\lambda)$ of exponential type $\sigma > 0$ is said to be of *sine-type* if there exist positive constants h, m, and M such that for $|\mathrm{Im}\ \lambda| \geq h$ the inequalities $m \leq |\omega(\lambda)|e^{-\sigma|\mathrm{Im}\ \lambda|} \leq M$ are satisfied.

Theorem 3.14. *Assume Hypothesis 3.1. Then the function $E_0(\lambda)$ given in (3.1) belongs to $\overline{SHB}_c$.*

Proof. Consider (1.1)–(1.6) with the potentials $q_3(x) = 0$, $x \in [0, \infty)$, and $q_1(x) = q_1^{(0)}(x) - c$ and $q_2(x) = q_2^{(0)}(x) - c$, where c is a real parameter independent on x and selected such that the operator A_0 corresponding to the "shifted" potentials $q_j^{(0)}(x)$, $j = 1, 2$, is strictly positive. In this case the spectrum of problem (3.12) is real. Therefore, according to assertion (vi) of Theorem 3.3, the spectrum of problem (3.4)–(3.7) lies in the closed upper half-plane. Temporarily denote by $E_0^{(0)}(-\lambda)$ the function computed by (3.1) but with $q_j(x)$ replaced by $q_j^{(0)}(x)$. We will prove first that $E_0^{(0)}(-\lambda) \in SHB$. It follows from Lemma 3.4 that $\lambda E_0^{(0)}(-\lambda)$ is a sine-type function. Then this function can be represented as $\lambda E_0^{(0)}(-\lambda) = \lambda C \lim_{n\to\infty} \prod_{k=-n}^{n} (1 - \lambda/\lambda_k)$, cf. [28, p.88]. Now by Theorem 6 of [26, Chap.VII] we obtain that $E_0^{(0)}(-\lambda) \in \overline{HB}$ (we can not claim that this function belongs to HB because it may have zeros on the real axis). Moreover, due to the symmetry $E_0^{(0)}(-\overline{\lambda}) = \overline{E_0^{(0)}(\lambda)}$, cf. (3.2), we conclude that $E_0^{(0)}(-\lambda) \in \overline{SHB}$. Next, passing to the case $c \neq 0$, we rewrite (3.4) as

$$y_j'' + (\lambda^2 + c - q_j^{(0)}(x))y_j = 0, \quad j = 1, 2, \tag{3.27}$$

and notice that the function $E_0(-\lambda)$, corresponding to problem (3.27) with the boundary conditions (3.5)–(3.7), belongs to $\overline{SHB}_c$. □

Introduce the following "even" and "odd" parts of the function $E_0(-\lambda)$:

$$\varphi_e(\lambda) = \Big(E_0(\lambda) + E_0(-\lambda)\Big)/2,\ \varphi_o(\lambda) = \Big(E_0(-\lambda) - E_0(\lambda)\Big)/(2i), \tag{3.28}$$

$$\hat{\varphi}_o(\lambda) = \lambda^{-1}\varphi_o(\lambda), \quad \lambda \in \mathbb{C}. \tag{3.29}$$

Due to (3.2), it follows that the functions $\varphi_e(\lambda)$ and $\varphi_o(\lambda)$ are real-valued for $\lambda \in \mathbb{R}$. Let us denote by $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$ the set of zeros of the function $\varphi_e(\lambda)$ and by $\{\theta_k\}_{-\infty,k\neq 0}^{\infty}$ the set of zeros of the function $\lambda^{-1}\varphi_o(\lambda)$. The enumeration is symmetric with respect to the origin, i.e. $\mu_{-k} = -\mu_k$, $\mu_k^2 < \mu_{k+1}^2$ and $\theta_{-k} = -\theta_k$, $\theta_k^2 \leq \theta_{k+1}^2$. We recall notation F_j, $j = 1, 2$, from Lemma 3.4.

Theorem 3.15. *Assume Hypothesis 3.1. Then:*

1. *All zeros μ_n and θ_n are simple, and either real or pure imaginary.*
2. *For every $k > 1$ either $\theta_{k-1}^2 < \mu_k^2 < \theta_k^2$ or $\mu_{k-1}^2 < \theta_{k-1}^2 = \mu_k^2 = \theta_k^2 < \mu_k^2$.*

(3) *The sequence* $\{\mu_k\}_{k=-\infty,k\neq 0}^{\infty}$ *has the following asymptotic behavior:*

$$\mu_k = \frac{\pi k}{2a} + \frac{F_1+F_2}{\pi k} + \frac{\gamma_k}{k}, \quad as \quad |k|\to\infty, \tag{3.30}$$

where $\{\gamma_k\}_{-\infty,k\neq 0}^{\infty}$ *is a sequence from* ℓ_2.

(4) *The sequence* $\{\theta_k\}_{-\infty,k\neq 0}^{\infty}$ *can be split in two subsequences so that*

$$\{\theta_k\}_{-\infty,k\neq 0}^{\infty} = \{\nu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty} \cup \{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$$

and the following asymptotic relations hold:

$$\nu_k^{(1)} = \frac{\pi k}{a} + \frac{F_1}{\pi k} + \frac{\beta_k^{(1)}}{k}, \quad as \quad |k|\to\infty, \tag{3.31}$$

$$\nu_k^{(2)} = \frac{\pi k}{a} + \frac{F_2}{\pi k} + \frac{\beta_k^{(2)}}{k}, \quad as \quad |k|\to\infty, \tag{3.32}$$

where $\{\beta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ *is a sequence from* ℓ_2, $j=1,2$.

Proof. Substituting (3.1) in (3.28)–(3.29), we infer:

$$\varphi_e(\lambda) = s_1'(\lambda,a)s_2(\lambda,a) + s_2'(\lambda,a)s_1(\lambda,a), \tag{3.33}$$

$$\hat\varphi_o(\lambda) = s_1(\lambda,a)s_2(\lambda,a). \tag{3.34}$$

The functions $\varphi_e(\lambda)$ and $\hat\varphi_o(\lambda)$ are real, and therefore all μ_k and θ_k are real or pure imaginary. The set of zeros of $\varphi_e(\lambda)$ coincides with the spectrum of problem (3.12), or, which is the same, with the spectrum of the problem

$$y_j'' + \lambda^2 y_j - q_j(x)y = 0, \quad y_j(\lambda,0)=0,\ j=1,2,$$
$$y_1(\lambda,a) = y_2(\lambda,a), \quad y_1'(\lambda_a) + y_2'(\lambda,a) = 0,$$

and, using [30, Thm.3.4.1], we obtain (3.30). Similarly, (3.34) implies (3.31)-(3.32).

Due to Theorem 3.14, the function $E_0(-\lambda) = \varphi_e(\lambda) + i\lambda\hat\varphi_o(\lambda)$ belongs to $\overline{SHB}_c$. Then $E_0(-\lambda) = \tilde\varphi_e(\lambda^2) + i\lambda\tilde\varphi_o(\lambda^2)$ where we define $\tilde\varphi_e(\lambda^2) = \varphi_e(\lambda)$ and $\tilde\varphi_o(\lambda^2) = \hat\varphi_o(\lambda)$. Clearly, there exists a constant $c>0$ such that the function $\tilde\varphi_e(\lambda^2-c) + i\lambda\tilde\varphi_o(\lambda^2-c)$ belongs to $\overline{SHB}$, and according to Lemma 3.11, we have $\tilde\varphi_e(\lambda-c) + i\tilde\varphi_o(\lambda-c) \in \overline{HB}$. Thus, we can apply Theorem 3' in [26, Sec.VII.2], cf. also Appendix B, and obtain the inequality

$$\cdots \le \theta_{k-1}^2 \le \mu_k^2 \le \theta_k^2 \le \mu_{k+1}^2 \le \cdots. \tag{3.35}$$

If $\mu_k = \theta_n$ for some $k\neq 0$ and n, that is, if $\varphi_e(\theta_n) = \varphi_o(\theta_n) = 0$, then either $s_1(\theta_n,a)=0$ or $s_2(\theta_n,a)=0$. Suppose that $s_1(\theta_n,a)=0$; then from (3.33) we obtain $s_1'(\theta_n,a)s_2(\theta_n,a)=0$. Consequently, $s_2(\theta_n,a)=0$, and θ_n is a double zero. Assertions (1) and (2) follow. Now assertion (3) follows from the fact that μ_k are the eigenvalues of problem (3.12). Statement (4) follows from the fact that the set of zeros of $s_j(\lambda,a)$ coincides with the spectrum of the Dirichlet problem $y_j'' + \lambda^2 y_j - q_j(x)y_j = 0$, $y_j(\lambda,0) = y_j(\lambda,a) = 0$. □

Lemma 3.16. *Assume Hypothesis 3.1. The function* $E_0(-\lambda)$ *can be represented as*

$$E_0(\lambda) = \Big(g_1(\lambda)(3g_2(\lambda) - g_2(-\lambda)) - g_1(-\lambda)(g_2(\lambda) - g_2(-\lambda))\Big)/(4i\lambda), \tag{3.36}$$

where the functions $g_j(\lambda)$ *belong to* SHB_c, *and are given by*

$$g_j(\lambda) = e^{i\lambda a}\left(1 - iF_j\lambda^{-1} + \xi_j(\lambda)\lambda^{-1}\right) \text{ with some } \xi_j(\lambda)\in\mathcal{L}^a, j=1,2. \tag{3.37}$$

Proof. For $j = 1, 2$ we denote

$$g_j(\lambda) = s_j'(\lambda, a) + i\lambda s_j(\lambda, a). \tag{3.38}$$

Then, substituting $s_j'(\lambda, a) = (g_j(\lambda)+g_j(-\lambda))/2$, $s_j(\lambda, a) = (g_j(\lambda)-g_j(-\lambda))/(2i\lambda)$ in (3.1), we obtain (3.36). Representation (3.37) follows by substituting (3.14) and (3.15) in (3.38). It is well known that the squares of zeros $(\nu_k^{(j)})^2$ of the functions $s_j(\lambda, a)$ interlace with the squares of zeros $(\gamma_k^{(j)})^2$ of the function $s_j'(\lambda, a)$ in the following strict sense (see [30, Sec.3.4]): $(\gamma_1^{(j)})^2 < (\nu_1^{(j)})^2 < (\gamma_2^{(j)})^2 < \dots$. Therefore, there exists a constant c such that for the zeros of the functions $s_j'(\sqrt{\lambda^2 - c}, a)$ and $s_j(\sqrt{\lambda^2 - c}, a)$ the following inequalities hold:

$$... - \sqrt{(\nu_{-1}^{(j)})^2 + c} < -\sqrt{(\gamma_{-1}^{(j)})^2 + c} < 0 < \sqrt{(\gamma_1^{(j)})^2 + c} < \sqrt{(\nu_1^{(j)})^2 + c} < ....$$

Also, we know from [30] that

$$s_j(\lambda, a) = a \prod\nolimits_{k=1}^{\infty} \frac{a^2}{\pi^2 k^2}(\nu_k^2 - \lambda^2) \quad \text{and} \quad s_j'(\lambda, a) = \prod\nolimits_{k=1}^{\infty} \Big(\frac{2a}{(2k-1)\pi}\Big)^2 (\gamma_k^2 - \lambda^2).$$

Therefore,

$$s_j(\sqrt{\lambda^2 - c}, a) = a \prod\nolimits_{k=1}^{\infty} \frac{a^2}{\pi^2 k^2}(\nu_k^2 + c - \lambda^2),$$
$$s_j'(\sqrt{\lambda^2 - c}, a) = \prod\nolimits_{k=1}^{\infty} \Big(\frac{2a}{(2k-1)\pi}\Big)^2 (\gamma_k^2 + c - \lambda^2).$$

Thus, the function $s_j'(\sqrt{\lambda^2 - c}, a)+i\lambda s_j(\sqrt{\lambda^2 - c}, a)$ satisfies the conditions of Corollary B.3 and therefore belongs to the class HB. Since it is symmetric, it also belongs to SHB. As a result, we have the inclusion $g_j(\lambda) \in SHB_c$. □

4. Inverse problem

In this section we consider the problem of recovering the potentials $q_1(x)$ and $q_2(x)$ from scattering data assuming that the potential $q_3(x)$ is identically equal to zero on the semi-infinite part of the wave-guide. In fact, in Theorems 4.3 and 4.4 we show how to recover the potentials as soon as we are given a function $E_0(\lambda)$ with the properties similar to the properties of the Jost function discussed in the previous sections.

Before proceeding with the solution of the inverse problem when $E_0(-\lambda)$ is given, we make the following remark: Even in the case when $q_3(x) = 0$, $x \in [0, \infty)$, the Jost function $E_0(-\lambda)$ is *not* uniquely determined by the scattering function $S(\lambda)$ as long as $E_0(\lambda)$ is allowed to have zeros on the real axis or pairs of pure imaginary zeros symmetric about the real axis. To illustrate this, let us suppose the λ_k is a real zero of $E_0(\lambda)$. Then, due to the symmetry (3.2), $-\lambda_k$ is also a zero of $E_0(\lambda)$, and $\pm\lambda_k$ are zeros of $E_0(-\lambda)$ as well. Cancellation of the corresponding factors in the fraction $S(\lambda) = E_0(\lambda)/E_0(-\lambda)$ shows that the scattering function $S(\lambda)$ does not change as long as we move zeros of $E_0(\lambda)$ along the real axis in a symmetric fashion. Similarly, we can achieve the same cancellation effect if we suppose that $E_0(\lambda)$ has two symmetrically located pure imaginary zeros $\lambda_k = i|\lambda_k|$ and $\lambda_{-k} = -i|\lambda_k|$; indeed, in this case we can move λ_k and λ_{-k} along the imaginary axis preserving the symmetry $|\lambda_k| = |\lambda_{-k}|$ and having $S(\lambda)$ unchanged. However, if we exclude these possibilities, that is, if we assume *a'priori* that $E_0(-\lambda)$ may have only a single simple zero at the origin and does not have any other real zeros nor any pairs of symmetric about the origin pure imaginary zeros, then the Jost function is uniquely determined by the scattering function. Indeed, the S-function is meromorphic due to $q_3(x) = 0$, $x \in [0, \infty)$, and, under the *a'priori* assumptions above, it is clear that the zeros of $S(\lambda)$ are the zeros of $E_0(\lambda)$, see (3.3). The function $\lambda E_0(-\lambda)$ is a sine-type function and therefore, see [28], the set of its zeros together with their asymptotics uniquely determine $E_0(\lambda)$, cf. Corollary 3.8.

Passing to the solution of the inverse problem, we will now describe the properties of a meromorphic function $S(\lambda)$ that enable us to construct a function $E_0(\lambda)$ having the same properties as the Jost function. Given a function $S(\lambda)$, we define, cf. (3.22), the function $\widehat{S}(\lambda)$ by the formula

$$\widehat{S}(\lambda) = S(\lambda) \cdot \frac{2\cos\lambda a + i\sin\lambda a}{2\cos\lambda a - i\sin\lambda a}.$$

Hypothesis 4.1. Assume that $S(\lambda)$ is a meromorphic in $\mathbb{C}$ function that satisfies the following conditions:

(a) $S(-\lambda) = 1/S(\lambda)$ and $S(-\overline{\lambda}) = \overline{S(\lambda)}$ for $\lambda \in \mathbb{C}$.

(b) $\left|\widehat{S}(\lambda) - 1\right| = O(|\lambda|^{-1})$ as $\lambda \to \pm\infty$, $\lambda \in \mathbb{R}$.

Proposition 4.2. *Assume that the function $S(\lambda)$ satisfies Hypothesis 4.1 and let Λ denote the set of poles of $S(\lambda)$. In addition, assume that one of the following conditions hold:*

(i) *If $S(0) = 1$ then the set $\Lambda = \{\lambda_k\}_{k=-\infty,k\neq 0}^{\infty}$ has properties (1)-(7) of Theorem 3.7 for $\Lambda^{(1)}$, and satisfies the asymptotic relations (3.23) with $\lambda_{2k-1}^{(0)}$ and $\lambda_{2k}^{(0)}$ given in formula (3.24).*

(ii) *If $S(0) = -1$ then the set $\Lambda \cup \{0\} = \{\lambda_k\}_{k=-\infty,k\neq 0}^{\infty}$ has properties (1)-(7) of Theorem 3.7 for $\Lambda^{(1)}$, and satisfies relations (3.23)-(3.24).*

Then there exists a unique entire function $E_0(\lambda)$ of exponential type $2a$ which has no real zeros (except, maybe, a simple zero at the origin), has no pairs of symmetric about the origin pure imaginary zeros, and satisfies the relations $S(\lambda) = E_0(\lambda)/E_0(-\lambda)$ and $\left|E_0(-\lambda) - \lambda^{-1}(\sin 2\lambda a + i\sin^2\lambda a)\right| = O(|\lambda|^{-2})$ as $\lambda \to \pm\infty$. In addition, $E_0(-\lambda) \in SHB_c$.

Proof. We know that $S(\lambda)$ as a meromorphic function, and thus we know its zeros and poles. Let us denote the poles of $S(\lambda)$ by λ_k. Then in case (i) we define $E_0(-\lambda)$ as the product $E_0(-\lambda) = \prod_{k=-\infty,k\neq 0}^{\infty}(1 - \lambda/\lambda_k)$. Using assertion (b) in Hypothesis 4.1, we conclude that $\lambda E_0(-\lambda)$ is a sine-type function, because the function $\lambda\prod_{k=-\infty,k\neq 0}^{\infty}(1-\lambda/\lambda_k^{(0)}) = C(2a)^{-1}(\sin 2\lambda a + i\sin^2\lambda a)$, where C is a constant, is a sine-type function (see [28]). In case (ii) we define $E_0(-\lambda) = \lambda\prod_{\lambda_k\in\Lambda}(1 - \lambda/\lambda_k)$, and conclude again that $\lambda E_0(-\lambda)$ is a sine-type function. The proof of the statement $\Phi(\lambda) \in SHB_c$ is similar to the proof of Propositions 4.8 and 4.9 in [41]. □

Next, we will discuss the solution of the inverse problem of recovering the potentials for the boundary value problem (1.1)–(1.2) with $q_3(x) = 0$, $x \in [0, \infty)$, given a function $E_0(-\lambda)$. Let us consider the following set of triples of real-valued potentials: $\mathcal{Q} = \{(q_j(x))_{j=1}^3 : q_j(x) \in L_2(0, a), j = 1, 2, q_3(x) = 0, x \in [0, \infty)\}$.

Theorem 4.3. *Assume that $E_0(-\lambda)$ is an entire function of exponential type $2a$ which satisfies the following conditions:*

(1) *$E_0(-\lambda)$ can be represented in form (3.36), where $g_j(\lambda) \in SHB_c$, j=1,2;*

(2) *$g_j(\lambda)$ can be represented in form (3.37), where F_j are some real constants and $\xi_j(\lambda)\in\mathcal{L}^a$, $j=1,2$.*

Then there exists a unique triple $(q_j(x))_{j=1}^3\in\mathcal{Q}$ such that the S-function for the boundary value problem (1.1)-(1.6) with the potentials $q_j(x)$ is given by (3.3).

Proof. We will prove that there exist real-valued potentials $q_j(x)\in L_2(0,a)$, $j=1,2$, such that the function $g_j(\lambda)$ is the Jost function of the problem

$$y_j''+\left(\lambda^2-\hat q_j(x)\right)y_j=0,\quad x\in[0,\infty),\quad y_j(\lambda,0)=0,$$

where $\hat q_j(x)=q_j(a-x)$ if $x\in[0,a)$ and $\hat q_j(x)=0$ if $x\in[a,\infty)$. For this, let us introduce the functions

$$g_j^e(\lambda)=(g_j(\lambda)+g_j(-\lambda))/2,\quad g_j^o(\lambda)=(g_j(\lambda)-g_j(-\lambda))/(2i). \tag{4.1}$$

Substituting (3.37) in (4.1), we obtain

$$g_j^e(\lambda)=\cos\lambda a+F_j\lambda^{-1}\sin\lambda a+(\xi_j(\lambda)+\xi_j(-\lambda))/2, \tag{4.2}$$

$$g_j^o(\lambda)=\sin\lambda a-F_j\lambda^{-1}\cos\lambda a+(\xi_j(\lambda)-\xi_j(-\lambda))/(2i). \tag{4.3}$$

We denote by $\{\mu_k^{(j)}\}_{-\infty,k\neq0}^{\infty}$ the set of zeros of $g_j^e(\lambda)$ and by $\{\nu_k^{(j)}\}_{-\infty}^{\infty}$ the set of zeros of $g_j^o(\lambda)$. It follows from (4.3) and [30, Lem.3.4.2] applied for the interval $[0,a]$ that $\{\nu_k^{(j)}\}_{k=-\infty,k\neq0}^{\infty}$ satisfy (3.31)–(3.32), and from (4.2) and [30, Lem.3.4.2] that $\{\mu_k^{(j)}\}_{-\infty,k\neq0}^{\infty}$ have the following asymptotics:

$$\mu_k^{(j)}=\pi(k-1/2)/a-F_j(\pi k)^{-1}+k^{-1}\gamma_k^{(j)},\quad\text{as}\quad|k|\to\infty, \tag{4.4}$$

where $\{\gamma_k^{(j)}\}_{-\infty,k\neq0}^{\infty}\in\ell_2$. It follows from (4.2) and (4.3) that $g_j^e(\lambda)$ and $\lambda^{-1}g_j^o(\lambda)$ are even functions. The condition $g_j(\lambda)\in SHB_c$ means that there exists $c\in\mathbb{R}$ such that $\tilde g_j^e(\lambda^2-c)+i\lambda\tilde g_j^o(\lambda^2-c)\in SHB$, where $\tilde g_j^e(\lambda^2)=g_j^e(\lambda)$, $\tilde g_j^o(\lambda^2)=\lambda^{-1}g_j^o(\lambda)$. By Lemma 3.11 we obtain $g_j^e(\sqrt{\lambda^2-c})+ig_j^o(\sqrt{\lambda^2-c})\in HB$. By Theorem B.1, the zeros $\{\pm\sqrt{(\mu_k^{(j)})^2+c}\}_{k=1}^{\infty}$ of the function $\tilde g_j^e(\lambda^2-c)$ and the zeros $\{\pm\sqrt{(\nu_k^{(j)})^2+c}\}_{k=1}^{\infty}$ of the function $\tilde g_j^o(\lambda^2-c)$ interlace:

$$0<\sqrt{(\mu_1^{(j)})^2+c}<\sqrt{(\nu_1^{(j)})^2+c}<\sqrt{(\mu_2^{(j)})^2+c}<\dots.$$

Consequently,

$$(\mu_1^{(j)})^2<(\nu_1^{(j)})^2<(\mu_2^{(j)})^2<\dots. \tag{4.5}$$

Now the sequences $\{\mu_k^{(j)}\}_{-\infty,k\neq0}^{\infty}$ and $\{\nu_k^{(j)}\}_{-\infty,k\neq0}^{\infty}$ satisfy all conditions of Theorem 3.4.1 in [30]. By this theorem, there exists a unique pair of real-valued potentials $q_j(x)\in L_2(0,a)$, $j=1,2$, such that the set $\{\nu_k^{(j)}\}_{-\infty,k\neq0}^{\infty}$ coincides with the spectrum of the Dirichlet problem

$$y_j''+(\lambda^2-q_j(x))y_j=0,\ x\in[0,a],\ y_j(\lambda,a)=y_j(\lambda,0)=0,\quad j=1,2, \tag{4.6}$$

while $\{\mu_k^{(j)}\}_{-\infty,k\neq0}^{\infty}$ is the spectrum of the Dirichlet - Neumann problem

$$y_j''+(\lambda^2-q_j(x))y_j=0,\quad x\in[0,a],\quad y_j(\lambda,a)=y_j'(\lambda,0)=0,\quad j=1,2.$$

Due to (3.1), the triple $(q_1(x),q_2(x),q_3(x)\equiv0)$, just constructed, generates the S-function by formula (3.3). Uniqueness follows from [30, Thm.3.4.1]. □

The next theorem gives even more explicit sufficient conditions for the ratio $E_0(\lambda)/E_0(-\lambda)$ to be an S-function of problem (1.1)-(1.6) with $q_3(x)=0$, $x\in[0,\infty)$.

Theorem 4.4. *Let $E_0(-\lambda)$ be a given entire function of exponential type $2a$ which satisfies the following conditions:*

(1) $E_0(-\lambda)\in SHB_c$;
(2) $E_0(-\lambda)$ *is of the form*

$$\begin{aligned} E_0(-\lambda) &= \lambda^{-1}\sin 2\lambda a - P_1\lambda^{-2}\cos 2\lambda a + P_2\lambda^{-3}\sin 2\lambda a + \phi_1(\lambda)\lambda^{-3} \\ &\quad + i\left(\lambda^{-1}\sin^2\lambda a - P_1/2\cdot\lambda^{-2}\sin 2\lambda a + P_3\lambda^{-3}\cos^2\lambda a + P_4\lambda^{-3}\sin^2\lambda a\right) \\ &\quad + i\phi_2(\lambda)\lambda^{-3}\sin\lambda a + i\phi_3(\lambda)\lambda^{-4}, \end{aligned} \tag{4.7}$$

where $P_k\in\mathbb{R}$, $k=1,\dots,4$, are given constants such that $P_1^2>4P_3$, and $\phi_l(\lambda)$ are given function such that $\phi_l(-\overline{\lambda})=\overline{\phi_l(\lambda)}$, $l=1,2,3$, and $\phi_1(\lambda),\phi_3(\lambda)\in\mathcal{L}^{2a}$, $\phi_2(\lambda)\in\mathcal{L}^{a}$.

Then there exists a triple $(q_1(x),q_2(x),q_3(x)\equiv 0)\in\mathcal{Q}$ such that the S-function for (1.1)-(1.6) is furnished by (3.3) with the given function $E_0(-\lambda)$.

Proof. Substituting (4.7) in (3.28), we compute:

$$\varphi_e(\lambda) = \lambda^{-1}\sin 2\lambda a - P_1\lambda^{-2}\cos 2\lambda a + P_2\lambda^{-3}\sin 2\lambda a + \psi_1(\lambda)\lambda^{-3}, \tag{4.8}$$

$$\begin{aligned} \varphi_o(\lambda) &= \lambda^{-1}\sin^2\lambda a - P_1/2\cdot\lambda^{-2}\sin 2\lambda a \\ &\quad + P_3\lambda^{-3}\cos^2\lambda a + P_4\lambda^{-3}\sin^2\lambda a + \psi_2(\lambda)\lambda^{-3}\sin\lambda a + \psi_3(\lambda)\lambda^{-4}. \end{aligned} \tag{4.9}$$

Let us denote by $\{\mu_k\}_{k=-\infty,k\neq 0}^{\infty}$ the set of zeros of the function $\varphi_e(\lambda)$ and by $\{\nu_k\}_{k=-\infty}^{\infty}$ the set of zeros of the function $\varphi_o(\lambda)$. Condition (1) of Theorem 4.4 implies that all zeros of the functions $\varphi_e(\lambda)$ and $\varphi_o(\lambda)$ are real or pure imaginary. We enumerate them in the following way: $\mu_{-k}=-\mu_k$, $(\mu_k)^2\le(\mu_{k+1})^2$, and $\nu_{-k}=-\nu_k$, $(\nu_k)^2<(\nu_{k+1})^2$ for $k\neq 0$, and $\nu_0=0$. An application of [37, Lem.2.1] shows that the sequence $\{\mu_k\}_{k=-\infty,k\neq 0}^{\infty}$ has the following asymptotic behavior:

$$\mu_k = \pi k/(2a) + P_1(\pi k)^{-1} + \gamma_k k^{-2}, \quad \text{as} \quad |k|\to\infty, \tag{4.10}$$

where $\{\gamma_k\}_{k=-\infty,k\neq 0}^{\infty}$ is a sequence from ℓ_2. The asymptotic behavior of the sequence $\{\nu_k\}_{k=-\infty,k\neq 0}^{\infty}$ is described next.

Proposition 4.5. *The sequence $\{\nu_k\}_{k=-\infty,k\neq 0}^{\infty}$ satisfies the relation $\nu_{-k}=-\nu_k$ for all $k\neq 0$, and can be represented as a union of two subsequences, $\{\nu_k^{(1)}\}_{k=-\infty,\ k\neq 0}^{\infty}$ and $\{\nu_k^{(2)}\}_{k=-\infty,\ k\neq 0}^{\infty}$, such that*

$$\nu_k^{(1)} = \frac{\pi k}{a} - \frac{F_1}{\pi k} + \frac{\beta_k^{(1)}}{k^2}, \quad \textit{as} \quad |k|\to\infty, \tag{4.11}$$

$$\nu_k^{(2)} = \frac{\pi k}{a} - \frac{F_2}{\pi k} + \frac{\beta_k^{(2)}}{k^2}, \quad \textit{as} \quad |k|\to\infty, \tag{4.12}$$

where $F_1=(-P_1+\sqrt{(P_1)^2-4P_3})/2$, $F_2=(-P_1-\sqrt{(P_1)^2-4P_3})/2$, and the sequence $\{\beta_k^{(j)}\}_{k=-\infty,k\neq 0}^{\infty}$ belongs to ℓ_2, $j=1,2$.

Proof. We consider $\lambda^{-1}\varphi_o(\lambda)$ as a perturbation of the following function:

$$\begin{aligned} \lambda^{-1}\varphi_o^{(0)}(\lambda) &= \lambda^{-1}\left(\lambda^{-1}\sin^2\lambda a - P_1\lambda^{-2}\sin\lambda a\cos\lambda a + P_3\lambda^{-3}\cos^2\lambda a\right) \\ &= \left(\lambda^{-1}\sin\lambda a + F_1\lambda^{-2}\cos\lambda a\right)\left(\lambda^{-1}\sin\lambda a + F_2\lambda^{-2}\cos\lambda a\right). \end{aligned} \tag{4.13}$$

Clearly, the set of zeros of this function can be split in two subsequences with the following asymptotic behavior:

$$\nu_k^{(01)} = \pi k/a - F_1(\pi k)^{-1} + \beta_k^{(01)} k^{-2},\ \nu_k^{(02)} = \pi k/a - F_2(\pi k)^{-1} + \beta_k^{(02)} k^{-2}$$

as $|k| \to \infty$, where $\{\beta_k^{(0j)}\}_{k=-\infty,k\neq 0}^{\infty} \in \ell_2$, $j = 1,2$. It follows that for any $\rho \in (0,(F_1 - F_2)/2)$ there exists a $k_1(\rho) \in \mathbb{N}$ such that for each $k > k_1(\rho)$ the disc of radius $\rho(\pi k)^{-1}$ centered at $\pi k/a - F_1(\pi k)^{-1}$ or $\pi k/a - F_2(\pi k)^{-1}$ contains exactly one simple zero of the function $\lambda^{-1}\varphi_o^{(0)}(\lambda)$. Let us introduce a variable $\tau = \tau(k,\rho,\theta)$ by the formula $\tau = \pi k/a - F_1(\pi k)^{-1} + \rho e^{i\theta}(\pi k)^{-1}$, where $\rho > 0$ and $\theta \in [0, 2\pi)$.

First, let us estimate $|\tau\varphi_0^{(0)}(\tau)|$ from below. We note that the inequalities

$$\left|\sin\tau a + (-1)^k \left((aF_1 + a\rho e^{i\theta})(\pi k)^{-1}\right)^2\right| \le C_{k_1}(\rho)\, k^{-3}, \quad k \in \mathbb{N},$$
$$\left|\cos\tau a - (-1)^k\right| \le C_{k_1} k^{-2}$$

hold uniformly with respect to $\theta \in [0, 2\pi)$ and $k > k_1 \in \mathbb{N}$ with a positive constant C_{k_1}. Therefore,

$$\left|\tau^{-1}\sin\tau a + F_1\tau^{-2}\cos\tau a - (-1)^k a^2\rho e^{i\theta}(\pi k)^{-2}\right| < \tilde{C}_{k_2} k^{-4}, \tag{4.14}$$

$$\left|\tau^{-1}\sin\tau a + F_2\tau^{-2}\cos\tau a - (-1)^k a^2(F_2 - F_1 + \rho e^{i\theta})(\pi k)^{-2}\right| < \tilde{C}_{k_2} k^{-4} \tag{4.15}$$

uniformly with respect to $\theta \in [0, 2\pi)$ for $k > k_2 \in \mathbb{N}$, where $\tilde{C}_{k_2} > 0$. Using (4.13)-(4.15), we obtain

$$\left|\tau^{-1}\varphi_0^{(0)}(\tau) - a^4\rho e^{i\theta}(F_2 - F_1 + \rho e^{i\theta})(\pi k)^{-2}/k^{-2}\right| < C_{k_3}^0 k^{-6} \tag{4.16}$$

for $k > k_3$. Since $|F_2 - F_1| > \rho$, we conclude that for some $\tilde{C}_{k_4}^0 > 0$ and $k > k_4$ the following inequality holds:

$$|\tau^{-1}\varphi_0^{(0)}(\tau)| > \tilde{C}_{k_4}^0 k^{-4}. \tag{4.17}$$

Next, let us estimate $|\tau^{-1}(\varphi_0(\tau) - \varphi_0^{(0)}(\tau))|$ from above:

$$\begin{aligned}|\tau^{-1}(\varphi_0(\tau) - \varphi_0^{(0)}(\tau))| &= \left|P_4\tau^{-4}\sin^2\tau a + \psi_2(\tau)\tau^{-3}\sin\lambda a + \psi_1(\tau)\tau^{-4}\right| \\ &< \beta_k(\rho)k^{-4},\end{aligned} \tag{4.18}$$

where $\beta_k \to 0$ as $k \to +\infty$. It follows from (4.18) that for any fixed $\rho < |F_2 - F_1|/2$ uniformly with respect to $\theta \in [0, 2\pi)$ there exists $C_{k_5} = C_{k_5}(\rho) \in (0, \tilde{C}_{k_4}^0)$ such that

$$|\tau^{-1}(\varphi_0(\tau) - \varphi_0^{(0)}(\tau))| < C_{k_5} k^{-4} \tag{4.19}$$

for $k > k_5$. Comparing (4.17) with (4.19) we obtain

$$|\tau^{-1}(\varphi_0(\tau) - \varphi_0^{(0)}(\tau))| < |\tau^{-1}\varphi_0^{(0)}(\tau)|. \tag{4.20}$$

Now Rouché Theorem implies that for $k > k_2(\rho)$ every disc of radius ρ centered at $\pi k/a - F_1/(\pi k)$ contains exactly one simple zero of the function $\lambda^{-1}\varphi_o(\lambda)$. We can choose ρ arbitrary small to achieve the following:

$$\nu_k^{(1)} = \pi k/a - F_1(\pi k)^{-1} + \kappa_k^{(1)} k^{-1}, \text{ where } \kappa_k^{(1)} = o(1) \text{ as } |k| \to \infty. \tag{4.21}$$

Similarly, we obtain

$$\nu_k^{(2)} = \pi k/a - F_2(\pi k)^{-1} + \kappa_k^{(2)} k^{-1}, \text{ where } \kappa_k^{(2)} = o(1) \text{ as } |k| \to \infty. \tag{4.22}$$

Let us substitute now (4.21) in the equation $(\nu_k^{(1)})^{-1}\varphi_o(\nu_k^{(1)})=0$ and make use of (4.10) and (4.13). Then we obtain:

$$\begin{aligned}\lambda^{-1}\varphi_o^{(0)}(\lambda) &= \Big((\nu_k^{(1)})^{-1}\sin\nu_k^{(1)}a + F_1(\nu_k^{(1)})^{-2}\cos\nu_k^{(1)}a\Big)\\ &\quad\times\Big((\nu_k^{(1)})^{-1}\sin\nu_k^{(1)}a + F_2(\nu_k^{(1)})^{-2}\cos\nu_k^{(1)}a\Big)\\ &\quad+\psi_2(\nu_k^{(1)})(\nu_k^{(1)})^{-4}\sin\nu_k^{(1)}a+\psi_1(\nu_k^{(1)})(\nu_k^{(1)})^{-5}=0.\end{aligned}\tag{4.23}$$

Substituting (4.21), we also have

$$\Big(\kappa_k^{(1)}a/(\pi k^2)+O(k^{-3})\Big)\Big((F_2-F_1)a^2(-1)^k(\pi k)^{-2}+O(k^{-3})\Big)=\beta_k^{(1)}k^{-5},$$

where $\{\beta_k^{(j)}\}_{k=-\infty}^{\infty}\in\ell_2$, yielding (4.11). The proof of (4.12) is similar. $\square$

Returning to the proof of Theorem 4.4, we remark that the function $E_0(-\lambda)$ belongs to SHB_c, and thus, using Lemma 3.11, the sequences $\{\mu_k\}_{k=-\infty,k\neq0}^{\infty}$ and $\{\nu_k\}_{k=-\infty,k\neq0}^{\infty}$ interlace in the following sense:

$$-\infty<(\mu_1)^2<(\nu_1)^2<(\mu_2)^2<(\nu_2)^2<\dots.\tag{4.24}$$

Adding a sufficiently large positive constant c to each $(\mu_k)^2$ and $(\nu_k)^2$ we obtain for $(\tilde\mu_k)^2=(\mu_k)^2+c$ and $(\tilde\nu_k)^2=(\nu_k)^2+c$ the inequalities $0<(\tilde\mu_1)^2<(\tilde\nu_1)^2<(\tilde\mu_2)^2<(\tilde\nu_2)^2<\dots$. Let us define $\tilde\mu_k$ and $\tilde\nu_k^{(j)}$ for $k=\pm1,\pm2,\dots$ as follows: $\tilde\mu_{\pm|k|}=\pm\sqrt{(\tilde\mu_{|k|})^2}$ and $\tilde\nu_{\pm|k|}^{(j)}=\pm\sqrt{(\tilde\nu_{|k|}^{(j)})^2}$, $j=1,2$. Due to (4.10), (4.11) and (4.12), these sequences have the following asymptotics:

$$\begin{aligned}\tilde\mu_k&=\pi k/(2a)+\tilde P_1(\pi k)^{-1}+\gamma_k k^{-2}, \quad \text{as} \quad |k|\to\infty,\\ \tilde\nu_k^{(1)}&=\pi k/a-\tilde F_1(\pi k)^{-1}+\beta_k^{(1)}k^{-2}, \quad \text{as} \quad |k|\to\infty,\\ \tilde\nu_k^{(2)}&=\pi k/a-\tilde F_2(\pi k)^{-1}+\beta_k^{(2)}k^{-2}, \quad \text{as} \quad |k|\to\infty,\end{aligned}$$

where $\tilde P_1=P_1-ca$, $\tilde F_j=F_j-ca/2$, $j=1,2$, and thus $\tilde P_1=\tilde F_1+\tilde F_2$, $\tilde F_1\neq\tilde F_2$.

Next, we observe that the three sequences, $\{\tilde\mu_k\}_{k-\infty,k\neq0}^{\infty}$, $\{\tilde\nu_k^{(1)}\}_{k=-\infty,k\neq0}^{\infty}$, and $\{\tilde\nu_k^{(2)}\}_{k=-\infty,k\neq0}^{\infty}$, satisfy conditions of Theorem 2.1 in [37]. For reader's convenience, this theorem is also recorded as Theorem B.4 in Appendix B of the current paper. Thus, using Theorem B.4, we conclude that there exists a unique pair of real potentials $\tilde q_1(x)$ and $\tilde q_2(x)$ such that the sequences $\{\tilde\mu_k\}_{k-\infty,k\neq0}^{\infty}$, $\{\tilde\nu_k^{(1)}\}_{k=-\infty,k\neq0}^{\infty}$, and $\{\tilde\nu_k^{(2)}\}_{k=-\infty,k\neq0}^{\infty}$ constitute, respectively, the spectra of the following three problems:

$$\begin{aligned}&y''+(\lambda^2-\tilde q(x))y=0,\ x\in[0,2a],\\ &y(\lambda,0)=y(\lambda,2a)=0;\end{aligned}\tag{4.25}$$

$$\begin{aligned}&y''+(\lambda^2-\tilde q_1(x))y=0,\ x\in[0,a],\\ &y(\lambda,0)=y(\lambda,a)=0;\end{aligned}\tag{4.26}$$

$$\begin{aligned}&y''+(\lambda^2-\tilde q_2(x))y=0,\ x\in[0,a],\\ &y(\lambda,0)=y(\lambda,a)=0.\end{aligned}\tag{4.27}$$

Here, $\tilde q(x)=\tilde q_1(x)$ for $x\in[0,a]$ and $\tilde q(x)=\tilde q_2(2a-x)$ for $x\in[a,2a]$. Consequently,

$$\tilde\varphi_e(\lambda)=2a\prod_{k=1}^{\infty}\Big(\frac{4a^2}{\pi^2k^2}\Big((\tilde\mu_k)^2-\lambda^2\Big)\Big)\tag{4.28}$$

is, in fact, equal to $s(\lambda, 2a)$, where $s(\lambda, x)$ is the solution of equation (4.25) that satisfies the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$ (indeed, (4.28) is a modification for the interval $[0, 2a]$ of formula (3.4.15) in [30]). Similarly, the expressions

$$\tilde{\varphi}_{01}(\lambda) = a\prod_{k=1}^{\infty}\left(\frac{a^2}{\pi^2k^2}\Big((\tilde{\nu}_k^{(1)})^2 - \lambda^2\Big)\right) \text{ and } \tilde{\varphi}_{02}(\lambda) = a\prod_{k=1}^{\infty}\left(\frac{a^2}{\pi^2k^2}\Big((\tilde{\nu}_k^{(2)})^2 - \lambda^2\Big)\right)$$

are equal, respectively, to $s_1(\lambda, a)$ and $s_2(\lambda, a)$, where $s_1(\lambda, x)$, respectively, $s_2(\lambda, x)$ is the solution of equation (4.26), respectively, equation (4.27) that satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$, $j = 1, 2$. According to (3.1), (3.33) and (3.34) this implies that $s(\lambda, 2a) = s_1'(\lambda, a)s_2(\lambda, a) + s_1(\lambda, a)s_2'(\lambda, a)$, that is, that the function $\tilde{E}_0(\lambda) = \tilde{\varphi}_e(\lambda) + i\lambda\tilde{\varphi}_{01}(\lambda)\tilde{\varphi}_{02}(\lambda)$ is the Jost function for the problem

$$\begin{aligned}
&y_j'' + (\lambda^2 - \tilde{q}_j(x))y_j = 0, \quad x \in [0, a], \quad j = 1, 2,\\
&y_3'' + \lambda^2 y_3 = 0, \quad x \in [0, \infty),\\
&y_1(\lambda, a) = y_2(\lambda, a) = y_3(\lambda, 0),\\
&y_1'(\lambda, a) + y_2'(\lambda, a) - y_3'(\lambda, 0) = 0,\\
&y_1(\lambda, 0) = y_2(\lambda, 0) = 0.
\end{aligned}$$

Now let us make the inverse transformation $(\tilde{\mu}_k)^2 \to (\mu_k)^2$, $(\tilde{\nu}_k^{(1)})^2 \to (\nu_k^{(1)})^2$, $(\tilde{\nu}_k^{(2)})^2 \to (\nu_k^{(2)})^2$. The corresponding functions are defined as follows:

$$\begin{aligned}
\varphi_e(\lambda) &:= 2a\prod\nolimits_{k=1}^{\infty}\left(\frac{4a^2}{\pi^2k^2}\Big((\mu_k)^2 - \lambda^2\Big)\right) = 2a\prod\nolimits_{k=1}^{\infty}\left(\frac{4a^2}{\pi^2k^2}\Big((\tilde{\mu}_k)^2 - c - \lambda^2\Big)\right),\\
\varphi_{01}(\lambda) &:= a\prod\nolimits_{k=1}^{\infty}\left(\frac{a^2}{\pi^2k^2}\Big((\nu_k^{(1)})^2 - \lambda^2\Big)\right) = a\prod\nolimits_{k=1}^{\infty}\left(\frac{a^2}{\pi^2k^2}\Big((\tilde{\nu}_k^{(1)})^2 - c - \lambda^2\Big)\right),\\
\varphi_{02}(\lambda) &:= a\prod\nolimits_{k=1}^{\infty}\left(\frac{a^2}{\pi^2k^2}\Big((\nu_k^{(2)})^2 - \lambda^2\Big)\right) = a\prod\nolimits_{k=1}^{\infty}\left(\frac{a^2}{\pi^2k^2}\Big((\tilde{\nu}_k^{(2)})^2 - \lambda^2\Big)\right).
\end{aligned}$$

Therefore, the function $E_0(\lambda) = \varphi_e(\lambda) + i\lambda\varphi_{01}(\lambda)\varphi_{02}(\lambda)$ is the Jost function for the boundary value problem (1.1)–(1.6) with $q_3(x) = 0$, $x \in [0, \infty)$. Thus, $q_j(x) = \tilde{q}_j(x) - c$, $j = 1, 2$, are the potentials that we had to construct in Theorem 4.4. □

Remark 4.6. Even if we assume *a'priori* that the Jost function $E_0(\lambda)$ has no real zeros (with a possible exception of a simple zero at the origin) and no pairs of pure imaginary zeros, then, given a function $S(\lambda)$, the choice of the pair of potentials $(q_j(x))_{j=1}^2$ is not unique because the choice of the sequences $\{\tilde{\nu}_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\tilde{\nu}_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ is not unique. However, as soon as the three spectra, $\{\tilde{\mu}_k\}_{-\infty,k\neq 0}^{\infty}$, $\{\tilde{\nu}_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$, and $\{\tilde{\nu}_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$, are fixed (and do not intersect), the procedure of recovering $\tilde{q}_1(x)$ and $\tilde{q}_2(x)$ from the three spectra, as described in [37], gives a unique pair $(\tilde{q}_j(x))_{j=1}^2$, and thus a unique pair $(q_j(x))_{j=1}^2$.

Appendix A

In this section we prove several abstract results from spectral theory of operator pencils mainly used in the proof of Theorem 3.3 (but also of some independent interest). First, we recall some terminology (for more details see, e.g., [31, Sec.11]).

Let $L(\lambda)$ be a pencil of linear operators acting on a separable complex Hilbert space H with the domain $D(L)$ independent of λ, and let $B(H)$ denote the set of bounded operators on H. The set $\varrho(L)$ of $\lambda \in \mathbb{C}$ such that $L(\lambda)^{-1} \in B(H)$ is called the *resolvent set* of the operator pencil $L(\lambda)$, and the set $\sigma(L) = \mathbb{C}\backslash\varrho(L)$ is called the *spectrum* of $L(\lambda)$. A number $\lambda_0 \in \mathbb{C}$ is called an *eigenvalue* of $L(\lambda)$ if there exists a nonzero vector $y_0 \in D(L)$ (called an *eigenvector*) such that $L(\lambda_0)y_0 = 0$. Nonzero vectors $y_1, y_2, \cdots, y_{p-1}$ are called *associated vectors* if

$$\sum_{k=0}^{n} \frac{1}{k!}\frac{d^k}{d\lambda^k} L(\lambda)\,|_{\lambda=\lambda_0}\; y_{n-k} = 0, \qquad n = 1, \cdots, p-1. \tag{A.1}$$

Here, the number p is called the *length of the chain* composed of the eigenvector and its associated vectors. An eigenvalue is called *semisimple* if it does not have associated vectors. The *geometric multiplicity* of an eigenvalue is defined as the maximal number of the corresponding linearly independent eigenvectors. The *algebraic multiplicity* is defined as the maximal value of the sum of the lengths of chains corresponding to the linearly independent eigenvectors. An eigenvalue is called *isolated* if it has a punctured neighborhood contained in the resolvent set. An isolated eigenvalue λ_0 of finite algebraic multiplicity is called *normal* if the subspace Im $L(\lambda_0)$ is closed. Let $\sigma_0(L)$ denote the set of normal eigenvalues of $L(\lambda)$.

In what follows we consider the quadratic operator pencil $L(\lambda) = \lambda^2 M - i\lambda K - A$ and, throughout, we assume that K and M are bounded operators on H, that is, $K, M \in B(H)$, and A is a closed operator with the domain $D(A)$ dense in H. As usual, the domain of the pencil is chosen to be $D(L(\lambda)) = D(M) \cap D(K) \cap D(A) = D(A)$, and is independent of λ.

Hypothesis A.1. Assume that:

(i) $M \geq 0$, $K \geq 0$, and $A = A^* \geq -\beta I$ for some positive β.
(ii) If $\beta_1 > \beta$ then the inverse operator $(A + \beta_1 I)^{-1}$ is a compact operator.
(iii) Ker $A \cap$ Ker $K \cap$ Ker $M = \{0\}$.

Under Hypothesis A.1, the spectrum of $L(\lambda)$ consists of normal eigenvalues only. This follows, in fact, from well-known results on analytic functions with values in the set of Fredholm operators on H, see, e.g., [15, Cor.XI.8.4]. Our first result is a generalization of Conclusion 2.4^0 in [23].

Theorem A.2. *Assume Hypothesis A.1. Then:*

(1) *If $A \geq 0$ then the spectrum of $L(\lambda)$ is located in the closed upper half-plane.*
(2) *If $A \geq \beta I$ for some $\beta > 0$, and $K > 0$ then the spectrum of $L(\lambda)$ is located in the open upper half-plane.*
(3) *If $A \geq \beta I$ for some $\beta > 0$, and $\lambda^2 My - Ay \neq 0$ for all real λ and all nonzero $y \in$ Ker K then the spectrum of $L(\lambda)$ is located in the open upper half-plane.*

Proof. Let $y_0 \neq 0$ be an eigenvector of $L(\lambda)$ corresponding to an eigenvalue λ_0. Then the equality $(L(\lambda_0)y_0, y_0) = 0$ implies:

$$((\text{Re } \lambda_0)^2 - (\text{Im } \lambda_0)^2)(My_0, y_0) + \text{Im } \lambda_0 (Ky_0, y_0) - (Ay_0, y_0) = 0, \tag{A.2}$$

$$\text{Re } \lambda_0 (2\text{Im } \lambda_0 (My_0, y_0) - (Ky_0, y_0)) = 0. \tag{A.3}$$

If Re $\lambda_0 \neq 0$, then $(My_0, y_0) \neq 0$ by *(iii)* in Hypothesis A.1, and the inequality Im $\lambda_0 \geq 0$ follows from (A.3) and *(i)* in Hypothesis A.1. If Re $\lambda_0 = 0$, then (A.2) implies Im $\lambda_0 (Ky_0, y_0) = (\text{Im } \lambda_0)^2 (My_0, y_0) + (Ay_0, y_0) \geq 0$ by *(i)* in Hypothesis A.1 and the assumption $A \geq 0$. Then $(Ky_0, y_0) \neq 0$ by *(iii)* in Hypothesis A.1, and thus Im $\lambda_0 \geq 0$, proving assertion (1).

Next, assume $A > \beta I$ for some $\beta > 0$ and $K > 0$, and suppose that Im $\lambda_0 = 0$ in (A.2) and (A.3). If Re $\lambda_0 = 0$, then (A.2) implies $(Ay_0, y_0) = 0$ in contradiction with positivity of A. If Re $\lambda_0 \neq 0$, then (A.3) implies $(Ky_0, y_0) = 0$, contrary to $K > 0$, proving assertion (2).

If Im $\lambda_0 = 0$ then (A.3) implies $y_0 \in \text{Ker } K$ because equality Re $\lambda_0 = 0$ by (A.2) contradicts positivity of A. But then $L(\lambda_0)y_0 = \lambda_0^2 My_0 - Ay_0 = 0$, in contradiction with the assumptions, proving (3). □

If A is not assumed to be nonnegative then $L(\lambda)$ might have eigenvalues in the open lower half-plane; they are located as follows.

Lemma A.3. *Assume Hypothesis A.1. Then:*

(1) *The part of the spectrum of $L(\lambda)$, located in the open lower half-plane, belongs to the imaginary axis.*
(2) *If $K > 0$ then the part of the spectrum of $L(\lambda)$, located in the closed lower half-plane, belongs to the imaginary axis.*

Proof. Let $y_0 \neq 0$ be an eigenvector of $L(\lambda)$ corresponding to an eigenvalue λ_0 with Im $\lambda_0 < 0$. Then for Re $\lambda_0 \neq 0$ equation (A.3) implies $(My_0, y_0) = (Ky_0, y_0) = 0$, and, consequently, $My_0 = Ky_0 = 0$. Then $L(\lambda_0)y_0 = Ay_0 = 0$, contradicting (*iii*) in Hypothesis A.1 and thus proving assertion (1). If $K > 0$ then for Im $\lambda_0 \leq 0$ the equality Re $\lambda_0 = 0$ follows from (A.3) as above, proving assertion (2). □

Lemma A.4. *Assume Hypothesis A.1. Then:*

(1) *All nonzero eigenvalues of $L(\lambda)$, located in the closed lower half-plane, are semisimple.*
(2) *If $K > 0$ on Ker A, then all eigenvalues of $L(\lambda)$, located in the closed lower half-plane, are semisimple.*

Proof. Let λ_0 be an eigenvalue of $L(\lambda)$ located in the open lower half-plane, let $y_0 \neq 0$ be a corresponding eigenvector, and suppose that there exists a nonzero associated vector y_1. Then, using (A.1), we compute:

$$\lambda_0^2 My_1 - i\lambda_0 Ky_1 - Ay_1 + 2\lambda_0 My_0 - iKy_0 = 0. \tag{A.4}$$

Multiplying (A.4) by y_0 we infer:

$$((\lambda_0^2 M - i\lambda_0 K - A)y_1, y_0) + ((2\lambda_0 M - iK)y_0, y_0) = 0. \tag{A.5}$$

Since λ_0 is pure imaginary by Lemma A.3, we have from (A.5):

$$(y_1, (\lambda_0^2 M - i\lambda_0 K - A)y_0) + ((2\lambda_0 M - iK)y_0, y_0) = 0, \tag{A.6}$$

which implies, taking the imaginary part, that

$$((2\text{Im } \lambda_0 M - K)y_0, y_0) = 0. \tag{A.7}$$

Now Im $\lambda_0 < 0$ implies $(My_0, y_0) = (Ky_0, y_0) = 0$, yielding $My_0 = Ky_0 = 0$. In this case $L(\lambda_0)y_0 = -Ay_0 = 0$ and, consequently, $y_0 \in \text{Ker } M \cap \text{Ker } K \cap \text{Ker } A$. Then, due to (*iii*) in Hypothesis A.1, we have $y_0 = 0$, a contradiction.

Next, let $\lambda_0 \neq 0$ be a real eigenvalue of $L(\lambda)$. Then (A.3) implies $(Ky_0, y_0) = 0$, and, consequently, $Ky_0 = 0$ and $(\lambda_0^2 M - A)y_0 = 0$. Then (A.6) implies $2\lambda_0(My_0, y_0) = 0$, yielding $My_0 = 0$. Hence, using $Ky_0 = 0$, we obtain $Ay_0 = 0$, which contradicts (*iii*) in Hypothesis A.1 again, proving assertion (1).

To prove assertion (2), we need to show that if $\lambda_0 = 0$ is an eigenvalue then it is semisimple. But if $\lambda_0 = 0$ then $y_0 \in \operatorname{Ker} A$ and (A.4) can be written as $Ay_1 + iKy_0 = 0$. Multiplying this by y_0, we have

$$(Ay_1, y_0) + i(Ky_0, y_0) = (y_1, Ay_0) + i(Ky_0, y_0) = i(Ky_0, y_0) = 0,$$

in contradiction with $K > 0$ on Ker A. □

Sometimes, it is more convenient to deal with bounded operator pencils. Assuming $A \geq -\beta I > -\beta_1 I$ for some positive β, we introduce the auxiliary bounded operator pencil $\tilde{L}(\lambda) = L(\lambda)(A + \beta_1 I)^{-1}$. The next lemma follows form [31, Lem.20.1].

Lemma A.5. *If $A \geq -\beta I > -\beta_1 I$ for some $\beta > 0$ then $\sigma(\tilde{L}(\lambda)) = \sigma(L(\lambda))$.*

Next, we introduce the family $L(\lambda, \eta) = \lambda^2 M - i\lambda\eta K - A$ of operator pencils depending on a parameter $\eta \in \mathbb{C}$ so that $L(\lambda, 1) = L(\lambda)$. Lemma A.5 enables us to use for the unbounded operator pencil $L(\lambda, \eta)$ the results of [10] (see also [17, 24]) established for bounded operator pencils. Adapted to the current discussion, these results can be summarized as follows.

Theorem A.6. *Assume Hypothesis A.1. Given $\eta_0 \in \mathbb{C}$, let Ω be a connected domain in $\mathbb{C}$ containing only one eigenvalue λ_0 of the pencil $L(\lambda, \eta_0)$. Let y_{l0}, $l = 1, \dots, \ell$, denote linearly independent eigenvectors corresponding to the eigenvalue λ_0 of the pencil $L(\lambda, \eta_0)$, and let p_l, $l = 1, \dots, \ell$, denote the length of the chain composed of the eigenvector y_{l0} and its associated vectors. Finally, let m denote the algebraic multiplicity of the eigenvalue λ_0. Then there exist numbers $\epsilon > 0$ and $m_0 \in \mathbb{N}$ such that $m_0 \leq m$ and for each η from the neighborhood $\{\eta \in \mathbb{C} : |\eta - \eta_0| < \epsilon\}$ of η_0 the following assertions hold:*

(1) *$L(\lambda, \eta)$ has exactly m_0 different eigenvalues in the domain Ω. These eigenvalues can be arranged in groups $\lambda_{lj}(\eta)$, where $j = 1, \dots, p_l$ and $l = 1, \dots, \ell$ such that $\sum_{l=1}^{\ell} p_l = m_0$. The groups can be chosen in the way that the functions $\lambda_{l1}(\eta)$, $\lambda_{l2}(\eta)$,…,$\lambda_{lp_l}(\eta)$, that belong to the same group, correspond to the complete set of p_l branches of the multi-valued function η^{1/p_l}. Moreover, these eigenvalues can be represented as the following series:*

$$\lambda_{lj}(\eta) = \lambda_0 + \sum_{k=1}^{\infty} a_{lk}(((\eta - \eta_0)^{1/p_l})_j)^k, \quad j = 1, \dots, p_l, \tag{A.8}$$

where $a_{lk} \in \mathbb{C}$ and $((\eta - \eta_0)^{1/p_l})_j$, $j = 1, \dots, p_l$, denotes the j-th branch of the multi-valued function $(\eta - \eta_0)^{1/p_l}$.

(2) *A basis in the eigenspace corresponding to $\lambda_{lj}(\eta)$ can be chosen as follows:*

$$y_{lj}^{(q)}(\eta) = y_{l0}^{(q)} + \sum_{k=1}^{\infty} y_{lk}^{(q)}(((\eta - \eta_0)^{1/p_l})_j)^k, \; j = 1, \dots, p_l, \; q = 1, \dots, \alpha_l, \tag{A.9}$$

where α_l is the geometric multiplicity of the eigenvalue $\lambda_{lj}(\eta)$, and the vectors $y_{l0}^{(q)}$, $q = 1, \dots, \alpha_l$, belong to the eigenspace of $L(\lambda, \eta_0)$ corresponding to the eigenvalue λ_0.

It should be mentioned that this theorem is a generalization of a well-known theorem on expansions for analytic functions in multi-valued case, cf. [32, Thm.13.3.6].

Corollary A.7. *Suppose that assumptions of Theorem A.6 hold. If λ_0 is a semisimple eigenvalue of $L(\lambda, \eta_0)$, then formulae (A.8) and (A.9) assume the form*

$$\lambda_l(\eta) = \lambda_0 + \sum_{k=1}^{\infty} a_{lk}(\eta - \eta_0)^k, \quad l = 1, \dots, \ell, \tag{A.10}$$

$$y_l^{(q)}(\eta) = y_{l0}^{(q)} + \sum_{k=1}^{\infty} y_{lk}^{(q)}(\eta-\eta_0)^k, \quad q = 1, \dots, \alpha_l. \tag{A.11}$$

Lemma A.8. *Suppose that assumptions of Theorem A.6 hold. Let $\lambda_k(\eta)$ with $\lambda_k(0) = i\tau$, where $\tau \in \mathbb{R}$, be an eigenvalue of $L(\lambda, \eta)$. Then:*

(1) *Re $\dot\lambda_k(0) = 0$ and Im $\dot\lambda_k(0) \ge 0$, where "dot" denotes $d/d\eta$.*
(2) *If $\tau < 0$, then Re $\dot\lambda_k(\eta) = 0$ and Im $\dot\lambda_k(\eta) \ge 0$ for all $\eta \ge 0$.*
(3) *If 0 is an eigenvalue of $L(\lambda, \eta)$ for some $\eta \ge 0$, then it is an eigenvalue for all $\eta \ge 0$. The algebraic multiplicity of the zero eigenvalue for $\eta = 0$ is even, and if it is denoted by 2κ, then for all $\eta > 0$ the algebraic multiplicity of the zero eigenvalue is equal to κ.*

Proof. Let $\eta_0 \in [0, 1]$ and let λ_0 with Re $\lambda_0 = 0$ and Im $\lambda_0 < 0$ be an eigenvalue of $L(\lambda, \eta_0)$. Due to Lemma A.4 this eigenvalue is semi-simple. Then (A.8) and (A.9) can be written as (A.10) and (A.11). Taking the η-derivative in $L(\lambda_l(\eta), \eta) y_l^{(q)}(\eta) = 0$ and multiplying the resulting equation by $y_l^{(q)}$, we infer for $\eta = \eta_0$:

$$a_{l1} = \frac{i\lambda_0 (K y_{i0}^{(q)}, y_{i0}^{(q)})}{2\lambda_0 (M y_{l0}^{(q)}, y_{l0}^{(q)}) - i\eta_0 (K y_{l0}^{(q)}, y_{l0}^{(q)})} = \frac{i\tau (K y_{l0}^{(q)}, y_{l0}^{(q)})}{2\tau (M y_{l0}^{(q)}, y_{l0}^{(q)}) - \eta_0 (K y_{l0}^{(q)}, y_{l0}^{(q)})}.$$

It is clear that Re $a_{l1} = 0$ and Im $a_{l1} \ge 0$ for $\eta_0 = 0$ and for $\eta_0 \ge 0$ and $\tau < 0$. □

We recall that the *total algebraic multiplicity* of the part of the spectrum of $L(\lambda)$ in a domain Ω is defined as $\sum_{k=1}^{n} m_k$, where m_k, $k = 1, \dots, n$, are the algebraic multiplicities of all n eigenvalues located in Ω. The following fact is a consequence of Corollary A.7, Lemma A.8, Theorem A.2, Lemma A.3, and Lemma A.4.

Corollary A.9.

(1) *Assume that $M \ge 0$, $K \ge 0$ and $M + K > \beta I$ for some $\beta > 0$. Then the total algebraic multiplicity of the part of the spectrum of $L(\lambda)$ located in the open lower half-plane coincides with the total algebraic multiplicity (which is equal to the geometric multiplicity) of the negative spectrum of A.*
(2) *If, in addition, $K > 0$ then the total algebraic multiplicity of the part of the spectrum of $L(\lambda)$ located in the closed lower half-plane coincides with the total algebraic multiplicity of the nonnegative spectrum of A.*

This fact (under different assumptions) was proved in [35] and [36]; for other versions of this result see [2, 4, 18, 44].

Appendix B

The main objective of this section is to prove Lemma 3.11. We will use the following theorem (see Theorem 3 in [26, Sec.VII.2]).

Theorem B.1. *Assume that $\omega(\lambda) = P(\lambda) + iQ(\lambda)$, where $P(\lambda)$ and $Q(\lambda)$ are real entire functions, and suppose that*

$$P(\lambda) = Ae^{u(\lambda)}(\lambda - a_0) \prod_{k=-\infty, k\ne 0}^{\infty} (1 - \lambda/a_k) e^{p_k(\lambda/a_k)}, \quad u(0) = 0,$$
$$Q(\lambda) = Be^{v(\lambda)}(\lambda - b_0) \prod_{k=-\infty, k\ne 0}^{\infty} (1 - \lambda/b_k) e^{p_k(\lambda/b_k)}, \quad v(0) = 0$$

are their expansions in infinite products. Then the function $\omega(\lambda)$ belongs to the class HB if and only if the following conditions hold:

(a) *The zeros a_k and b_k of the functions $P(\lambda)$ and $Q(\lambda)$ interlace, that is:*

(B.1) $$b_k < a_k < b_{k+1}, \quad k = \pm 1, \pm 2, \dots, \quad \text{and} \quad a_{-1} < 0 < b_1.$$

(b) *The real entire functions $u(\lambda)$ and $v(\lambda)$ and the exponents $p_k(\lambda/a_k)$ and $p_k(\lambda/b_k)$ satisfy the condition*

$$u(\lambda) - v(\lambda) + \sum_{k=-\infty}^{\infty} (p_k(\lambda/a_k) - p_k(\lambda/b_k)) = 0.$$

(c) *The constants A and B have opposite signs.*

Remark B.2. We note a misprint in assertion (c) of Theorem 3 in [26, Sec.7.2], where the clause "same signs" should be replaced by the clause "opposite signs".

The proof of Theorem 3 in [26, Sec.7.2] also gives the following corollary.

Corollary B.3. *Assume that $\omega(\lambda) = P(\lambda) + iQ(\lambda)$, where $P(\lambda)$ and $Q(\lambda)$ are real entire functions having the following expansions into infinite products:*

$$P(\lambda) = Ae^{u(\lambda)} \prod_{k=1}^{\infty} (1 - \lambda/a_k) e^{p_k(\lambda/a_k)}, \quad u(0) = 0,$$
$$Q(\lambda) = Be^{v(\lambda)} \prod_{k=1}^{\infty} (1 - \lambda/b_k) e^{p_k(\lambda/b_k)}, \quad v(0) = 0.$$

Then $\omega(\lambda)$ belongs to the class HB if and only if the following conditions hold:

(a) *The zeros a_k and b_k of the functions $P(\lambda)$ and $Q(\lambda)$ interlace:*

(B.2) $$a_{k-1} < b_{k-1} < a_k < b_k, \quad k = 2, 3, \dots .$$

(b) *The entire real-valued functions $u(\lambda)$ and $v(\lambda)$ and the exponents $p_k(\lambda/a_k)$ and $p_k(\lambda/b_k)$ satisfy the condition*

$$u(\lambda) - v(\lambda) + \sum_{k=1}^{\infty} (p_k(\lambda/a_k) - p_k(\lambda/b_k)) = 0.$$

(c) *The constants A and B have the same sign.*

We are ready to prove Lemma 3.11.

Proof. Using the symmetry of the given function $\omega(\lambda) \in SHB$, we enumerate the zeros of $Q(\lambda)$ so that $b_0 = 0$, $b_{-k} = -b_k$; then $a_0 > 0$ due to (B.1). Changing the numeration of a_k by letting $\tilde{a}_k = a_k$ for $k < 0$ and $\tilde{a}_k = a_{k-1}$ for $k > 0$, and using the symmetry $\tilde{a}_{-k} = -\tilde{a}_k$, we obtain:

$$P(\lambda) = -a_0 Ae^{u(\lambda^2)} \prod_{k=1}^{\infty} (1 - \lambda^2/\tilde{a}_k^2) e^{p_k(\lambda^2/\tilde{a}_k^2)}, \quad u(0) = 0,$$
$$Q(\lambda) = B\lambda e^{v(\lambda^2)} \prod_{k=1}^{\infty} (1 - \lambda^2/b_k^2) e^{p_k(\lambda^2/b_k^2)}, \quad v(0) = 0,$$

so that the following statements hold: (1) The zeros $\tilde{a}_k$ and b_k interlace: $0 < \tilde{a}_1 < b_1 < \tilde{a}_2 < b_2 < \dots$; (2) The entire real-valued functions $u(\lambda)$ and $v(\lambda)$ and the exponents $p_k(\lambda^2/\tilde{a}_k)$ and $p_k(\lambda/b_k)$ satisfy the condition

$$u(\lambda^2) - v(\lambda^2) + 2\sum_{k=1}^{\infty} \big(p_k(\lambda^2/\tilde{a}_k) - p_k(\lambda^2/b_k)\big) = 0;$$

(3) The constants $-a_0 A$ and B have the same sign. Therefore, we infer

$$\tilde{P}(\lambda) = -a_0 Ae^{u(\lambda)} \prod_{k=-\infty, k\neq 0}^{\infty} (1 - \lambda/\tilde{a}_k^2) e^{p_k(\lambda/a_k^2)}, \quad u(0) = 0,$$
$$\tilde{Q}(\lambda) = Be^{v(\lambda)} \prod_{k=1}^{\infty} (1 - \lambda/b_k^2) e^{p_k(\lambda/b_k^2)}, \quad v(0) = 0,$$

and an application of Corollary B.3 concludes the proof of Lemma 3.11. □

The proof of the following theorem can be found in [37, Thm.2.1].

Theorem B.4. *Assume that* $\{\mu_k\}_{k=-\infty,k\neq 0}^{\infty}$, $\{\nu_k^{(1)}\}_{k=-\infty,k\neq 0}^{\infty}$, $\{\nu_k^{(2)}\}_{k=-\infty,k\neq 0}^{\infty}$ *are three sequences of real numbers satisfying the relations* $\mu_k < \mu_{k+1}$, $\nu_k < \nu_{k+1}$, $\nu_k^{(1)} < \nu_{k+1}^{(1)}$, $\mu_{-k} = \mu_k$, $\nu_{-k}^{(1)} = -\nu_k^{(2)}$, $\nu_{-k}^{(2)} = \nu_k^{(2)}$ *for* $k = 1, 2, \dots,$ *and having the asymptotic properties given in (3.30)–(3.32), where* P_1, F_1 *and* F_2 *are real constants satisfying the inequality* $F_1 \neq F_2$ *and equality* $F_1 + F_2 = P_1$. *Also, assume that* $\{\nu_k^{(1)}\}_{k=-\infty,k\neq 0}^{\infty} \cap \{\nu_k^{(2)}\}_{k=-\infty,k\neq 0}^{\infty} = \emptyset$, *and the squares of the elements of the sequences* $\{\mu_k\}_{k=-\infty,k\neq 0}^{\infty}$ *and* $\{\nu_k\}_{k=-\infty,k\neq 0}^{\infty} = \{\nu_k^{(1)}\}_{k=-\infty,k\neq 0}^{\infty} \cup \{\nu_k^{(2)}\}_{k=-\infty,k\neq 0}^{\infty}$ *interlace as follows:* $0 < (\mu_1)^2 < (\nu_1)^2 < (\mu_2)^2 < (\nu_2)^2 < \dots$. *Then there exists a unique real-valued potential* $q(x) \in L_2(0, 2a)$ *such that the three sequences* $\{\mu_k\}_{k=-\infty,k\neq 0}^{\infty}$, $\{\nu_k^{(1)}\}_{k=-\infty,k\neq 0}^{\infty}$ *and* $\{\nu_k^{(2)}\}_{k=-\infty,k\neq 0}^{\infty}$, *respectively, constitute the spectra of the three boundary value problems (3.12), (4.6) for* $j = 1$, *and (4.6) for* $j = 2$, *respectively, where the potentials* q_j *in (4.6) defined via this* $q(x)$ *by* $q_1(x) = q(x)$ *and* $q_2(x) = q(2a - x)$, $x \in [0, a]$.

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