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FINITE DIMENSIONAL FOUR SPECTRA INVERSE PROBLEM

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We find necessary and sufficient conditions for four finite sequences of positive numbers to be certain parts of spectra of the Dirichlet-Dirichlet, Dirichlet-Neumann, Neumann-Dirichlet and Neumann-Neumann boundary value problems generated by the same Stieltjes string recurrence relations.

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Получены условия на четыре последовательности положительных чисел, необходимые и достаточные для того, чтобы эти последовательности были частями спектров задач Дирихле-Дирихле, Дирихле-Неймана, Неймана-Дирихле и Неймана-Неймана, порожденных рекуррентными соотношениями одной стильтьесовской струны.

1. Introduction. In the theory of inverse Sturm-Liouville problem much attention is paid to the question what spectral data uniquely determine the potential. Such investigations were started in [1], [2] and continued by many authors (see [5], [3], [4] and references therein). In [8] it was shown that certain parts of four spectra of the Dirichlet-Dirichlet, Dirichlet-Neumann, Neumann-Dirichlet and Neumann-Neumann problems uniquely determine the potential, and were established necessary and sufficient conditions for four sequences to be parts of spectra of the four boundary value problems generated by the same potential. We prove an analogue of this result for finite dimensional problems of Stieltjes string (a massless thread bearing point masses, see [6]) small transverse vibrations. In this case in addition to the spectra one needs to have the length of the string as a given data to find the values of point masses and the lengths of the subintervals into which the point masses divide the string.

We consider the boundary value problems generated by the recurrence relations of a Stieltjes string with different boundary conditions at the endpoints. Dirichlet conditions describe fixed endpoints while Neumann conditions describe endpoints free to move in the direction orthogonal to the equilibrium position of the string. Thus we deal with four problems: the Dirichlet-Dirichlet problem with the Dirichlet conditions at both endpoints, the Dirichlet-Neumann problem with the Dirichlet condition at the left end and the Neumann condition at the right end, the Neumann-Dirichlet problem with the Neumann condition at the left endpoint and the Dirichlet condition at the right one, and the Neumann-Neumann problem with the Neumann conditions at both endpoints. It is well known that the spectra of Dirichlet-Dirichlet and Dirichlet-Neumann problems together with the total length of the string uniquely determine the values of the point masses and of the lengths of the subintervals into which the masses divide the string. It is clear that instead of the spectrum of the

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Dirichlet-Neumann problem, the Neumann-Dirichlet spectrum can be taken as a given data. It is easy to show that one can also use the Neumann-Dirichlet and Neumann-Neumann spectra together with the total mass of the string to find the values of point masses and of the lengths of the subintervals. However, knowledge of the Dirichlet-Neumann and the Neumann-Dirichlet spectra even together with the total length and the total mass are not sufficient to determine the values of the masses and of the subintervals. We show how to choose parts of the four spectra to solve the inverse problem, i.e. to determine the values of the point masses and of the subintervals. Also we give a method of recovering these data and show that the solution of such an inverse problem is unique.

In Section 2 we prove an auxiliary result about solution of a polynomial identity similar to that in [7]. In Section 3 we describe the four boundary value problems generated by the Stieltjes string recurrence relations. In Section 4 we present the main result.

2. Solution of polynomial identity. We start with an abstract result.

Theorem 1. *Let $P_1(z)$ and $P_2(z)$ be polynomials of degree $2n$ which satisfy the identity*

$$P_1(z) - P_2(z) = C, \tag{1}$$

where $C \neq 0$ is a constant. Let the value of $P_1(z)$ be known at a point $z = a$ which is not a zero of the polynomial, i.e. $P_1(a) = \widetilde{a} \neq 0$. Let $\{x_i\}_{i=1}^n$ ($x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$, for all i) be a part of the set of zeros of the polynomial $P_1(z)$ and $\{y_i\}_{i=1}^n$ ($y_i \neq y_j$ for $i \neq j$ and $y_i \neq 0$ for all i) a part of the set of zeros of the polynomial $P_2(z)$ be given such that

$$\{x_i\}_{i=1}^n \cap \{y_i\}_{i=1}^n = \varnothing.$$

Then $P_1(z)$ and $P_2(z)$ are determined uniquely.

Proof. Since $x_i \neq 0$ and $y_i \neq 0$ for all $i \in \{1, 2, \dots, n\}$, the polynomials $P_1(z)$ and $P_2(z)$ can be represented as

$$P_1(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{x_k}\right) G(z), \tag{2}$$

$$P_2(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{y_k}\right) D(z), \tag{3}$$

where $D(z)$ and $G(z)$ are polynomials of degree n.

Considering (1) at $z = x_i$, $i \in \{1, 2, \dots, n\}$ with account of (3) we obtain

$$D(x_i) = -\frac{C}{\prod\limits_{k=1}^{n} \left(1 - \frac{x_i}{y_k}\right)}. \tag{4}$$

At the point $z = a$ identity (1) gives

$$D(a) = \frac{\widetilde{a} - C}{\prod\limits_{k=1}^{n} \left(1 - \frac{a}{y_k}\right)}. \tag{5}$$

Thus, we can construct the Lagrange interpolating polynomial for $D(z)$ using the zeros $z = x_i$, $i \in \{1, 2, \dots, n\}$ of the polynomial $P_1(z)$ and the point $z = a$ as the nodes of interpolation and the values of $D(z)$ at the nodes given by (4) and (5)

$$D(z) = \sum_{i=1}^{n} D(x_i) \frac{z-a}{x_i-a} \prod_{j=1, j\neq i}^{n} \frac{z-x_j}{x_i-x_j} + D(a) \prod_{j=1}^{n} \frac{z-x_j}{a-x_j} = \tag{6}$$

$$= -\left(\sum_{i=1}^{n} \frac{C}{\prod\limits_{k=1}^{n} \left(1-\frac{x_i}{y_k}\right)} \frac{z-a}{x_i-a} \prod_{j=1, j\neq i}^{n} \frac{z-x_j}{x_i-x_j} - \frac{\widetilde{a}-C}{\prod\limits_{k=1}^{n} \left(1-\frac{a}{y_k}\right)} \prod_{j=1}^{n} \frac{z-x_j}{a-x_j} \right).$$

Therefore,

$$P_2(z) = -\prod_{k=1}^{n} \left(1-\frac{z}{y_k}\right) \left(\sum_{i=1}^{n} \frac{C}{\prod\limits_{k=1}^{n} \left(1-\frac{x_i}{y_k}\right)} \frac{z-a}{x_i-a} \prod_{j=1, j\neq i}^{n} \frac{z-x_j}{x_i-x_j} - \frac{\widetilde{a}-C}{\prod\limits_{k=1}^{n} \left(1-\frac{a}{y_k}\right)} \prod_{j=1}^{n} \frac{z-x_j}{a-x_j} \right). \tag{7}$$

In the same way we obtain

$$P_1(z) = \prod_{k=1}^{n} \left(1-\frac{z}{x_k}\right) \left(\sum_{i=1}^{n} \frac{C}{\prod\limits_{k=1}^{n} \left(1-\frac{y_i}{x_k}\right)} \frac{z-a}{y_i-a} \prod_{j=1, j\neq i}^{n} \frac{z-y_j}{y_i-y_j} + \frac{\widetilde{a}}{\prod\limits_{k=1}^{n} \left(1-\frac{a}{x_k}\right)} \prod_{j=1}^{n} \frac{z-y_j}{a-y_j} \right). \tag{8}$$

Since each solution $(P_1(z), P_2(z))$ of (1) can be obtained in this way, the solution is unique. □

Remark 1. Let for a certain s: $y_s = 0$, then in the same way we obtain

$$P_2(z) = -z \prod_{k=1,\ k\neq s}^{n} \left(1-\frac{z}{y_k}\right) \times$$

$$\times \left(\sum_{i=1}^{n} \frac{C}{x_i \prod\limits_{k=1,\ k\neq s}^{n} \left(1-\frac{x_i}{y_k}\right)} \frac{z-a}{x_i-a} \prod_{j=1, j\neq i}^{n} \frac{z-x_j}{x_i-x_j} - \frac{\widetilde{a}-C}{a \prod\limits_{k=1,\ k\neq s}^{n} \left(1-\frac{a}{y_k}\right)} \prod_{j=1}^{n} \frac{z-x_j}{a-x_j} \right).$$

In the case of $x_s = 0$ the corresponding changes must be done for $P_1(z)$.

3. Stieltjes string. Let us consider a Stieltjes string of length l bearing n point masses $m_1, m_2, \dots, m_n$ which divide the string into subintervals of lengths $l_0, l_1, \dots, l_n$. Transversal displacements $V_k(t)$ of the masses m_k satisfies the recurrence relations

$$\frac{V_k(t)-V_{k-1}(t)}{l_{k-1}} + \frac{V_k(t)-V_{k+1}(t)}{l_k} - m_k V_k''(t) = 0, \quad (k \in \{1, 2, \dots, n\}). \tag{9}$$

If the left endpoint is fixed then the Dirichlet boundary condition at the left one is

$$V_0(t) = 0, \tag{10}$$

the Dirichlet condition at the right endpoint is

$$V_{n+1}(t) = 0. \tag{11}$$

Substituting $V_k(t) = U_k e^{i\rho t}$ into (9)–(11) we obtain

$$\frac{U_k - U_{k-1}}{l_{k-1}} + \frac{U_k - U_{k+1}}{l_k} + m_k \lambda U_k = 0, \quad (k \in \{1, 2, \ldots, n\}), \tag{12}$$

$$U_0 = 0, \tag{13}$$

$$U_{n+1} = 0, \tag{14}$$

where U_k is the amplitude of the mass m_k, $\lambda = \rho^2$ is the spectral parameter.

The eigenvalues $\{\nu\}_{k=1}^n$ of problem (12)–(14) are simple positive (see [6]) and we enumerate them successively

$$0 < \nu_1 < \nu_2 < \ldots . < \nu_n. \tag{15}$$

They are the zeros of the polynomial $R_{2n}(\lambda)$. The polynomials $R_k(\lambda)$, $k \in \{1, \ldots, 2n\}$ were introduced in [6]. They are the solutions of the recurrence relations

$$R_{2k}(\lambda) = l_k R_{2k-1}(\lambda) + R_{2k-2}(\lambda), \; R_{2k-1}(\lambda) = R_{2k-3}(\lambda) - m_k \lambda R_{2k-2}(\lambda) \; (k \in \{1, 2, \ldots, n\}) \tag{16}$$

and initial conditions

$$R_0(\lambda) \equiv 1, \; R_{-1}(\lambda) = \frac{1}{l_0}. \tag{17}$$

If the right endpoint of the string is free to move in the direction orthogonal to the equilibrium position of the string, then we have the Neumann condition at the right endpoint $V_{n+1}(t) = V_n(t)$ or for amplitudes

$$U_{n+1} = U_n. \tag{18}$$

Thus, we have Dirichlet-Neumann problem (12), (13), (18) with the characteristic equation

$$R_{2n-1}(\lambda) = 0. \tag{19}$$

The zeros $\{\mu_k\}_{k=1}^n$ of the polynomial R_{2n-1} are interlaced (see [6], p. 340) with $\{\nu_k\}_{k=1}^n$

$$0 < \mu_1 < \nu_1 < \mu_2 < \ldots < \mu_n < \nu_n. \tag{20}$$

Now let us consider a Stieltjes string with the left endpoint free to move in the direction orthogonal to the equilibrium position of the string. Then small transversal vibrations of such string are described by equations (12), where $k \in \{2, 3, \ldots, n\}$. The Neuman condition for amplitudes at the left endpoint is

$$U_1 = U_0. \tag{21}$$

If the right endpoint is fixed then we have again Dirichlet condition (14), while for the free right endpoint we have Neumann condition (18).

Thus, equations (12) for $k \in \{2, 3, \ldots, n\}$, (14), (21) compose Neumann-Dirichlet spectral problem while equations (12) for $k \in \{2, 3, \ldots, n-1\}$, (18), (21) compose Neumann-Neumann spectral problem.

Following [6] we introduce polynomials $Q_k(z)$ by the recurrence relations

$$Q_{2k}(\lambda) = l_k Q_{2k-1}(\lambda) + Q_{2k-2}(\lambda),\ Q_{2k-1}(\lambda) = Q_{2k-3}(\lambda) - m_k\lambda Q_{2k-2}(\lambda) \tag{22}$$

and initial conditions

$$Q_0(\lambda) \equiv 1, \quad Q_{-1}(\lambda) \equiv 0. \tag{23}$$

The zeros $\{\kappa_k\}_{k=1}^n$ of the polynomial $Q_{2n}(\lambda)$ $(0 < \kappa_1 < \kappa < \ldots < \kappa_n)$ are the eigenvalues of the Neumann-Dirichlet spectral problem (12) for $k \in \{2, 3, \ldots, n\}$, (14), (21).

The zeros $\{\zeta_k\}_{k=0}^{n-1}$ $(0 = \zeta_0 < \zeta_1 < \ldots < \zeta_{n-1})$ of the polynomial $Q_{2n-1}(\lambda)$ are the eigenvalues of Neumann-Neumann spectral problem (12) for $k \in \{2, 3, \ldots, n-1\}$, (18), (21).

It is known also that $\{\zeta_k\}_{k=1}^n$ interlace with $\{\kappa_k\}_{k=1}^n$

$$0 = \zeta_0 < \kappa_1 < \zeta_1 < \kappa_2 < \zeta_2 < \kappa_3 < \ldots < \zeta_{n-1} < \kappa_n. \tag{24}$$

Also it should be mentioned that using $\{\mu_k\}_{k=1}^n$ and $\{\nu_k\}_{k=1}^n$ and the total length $l > 0$ and expending into continued fraction we can find all the masses and the subintervals

$$l\frac{\prod\limits_{k=1}^{n}\left(1-\frac{z}{\nu_k}\right)}{\prod\limits_{k=1}^{n}\left(1-\frac{z}{\mu_k}\right)} = l_n + \cfrac{1}{-m_n z + \cfrac{1}{l_{n-1} + \cfrac{1}{-m_{n-1}z + \cfrac{1}{\ldots + \cfrac{1}{l_0}}}}}. \tag{25}$$

Moreover, the following theorem is due to [6].

Theorem 2. *For any two sequences of positive numbers $\{\mu_k\}_{k=1}^n$ and $\{\nu_k\}_{k=1}^n$ which satisfy (20) and any positive number l there exists a unique couple of sequences $\{m_k\}_{k=1}^n$ and $\{l_k\}_{k=0}^n$ which generate problem (12)–(14) with the spectrum $\{\nu_k\}_{k=1}^n$ and problem (12), (13), (18) with the spectrum $\{\mu_k\}_{k=1}^n$.*

4. Main result. Now we are ready to state our main result.

Theorem 3. *Let four sequences of natural numbers $\{k_j\}_{j=1}^{n_1}$, $\{p_j\}_{j=1}^{n_2}$, $\{r_j\}_{j=1}^{n_3}$, $\{s_j\}_{j=1}^{n_4}$ be such that*

$k_j = k_{j'} \Leftrightarrow j = j'$, $p_j = p_{j'} \Leftrightarrow j = j'$, $\{k_j\}_{j=1}^{n_1} \cap \{p_j\}_{j=1}^{n_2} = \varnothing$, $\{k_j\}_{j=1}^{n_1} \cup \{p_j\}_{j=1}^{n_2} = \{1, 2, 3, \ldots, n-1\}$, $n_1 + n_2 = n - 1$

$r_j = r_{j'} \Leftrightarrow j = j'$, $s_j = s_{j'} \Leftrightarrow j = j'$, $\{r_j\}_{j=1}^{n_3} \cap \{s_j\}_{j=1}^{n_4} = \varnothing$, $\{r_j\}_{j=1}^{n_3} \cup \{s_j\}_{j=1}^{n_4} = \{1, 2, 3, \ldots, n\}$, $n_3 + n_4 = n$.

Let a constant $l > 0$ be given together with four sequences of positive numbers $\{\nu_{k_j}\}_{j=1}^{n_1}$, $\{\zeta_{p_j}\}_{j=1}^{n_2}$, $\{\mu_{r_j}\}_{j=1}^{n_3}$, $\{\kappa_{s_j}\}_{j=1}^{n_4}$ such that the enumerated successively the sequences

$$\{\eta_k\}_{k=1}^n = \{\nu_n\} \cup \{\nu_{k_j}\}_{j=1}^{n_1} \cup \{\zeta_{p_j}\}_{j=1}^{n_2} \tag{26}$$

and

$$\{\gamma_k\}_{k=1}^n = \{\mu_{r_j}\}_{j=1}^{n_3} \cup \{\kappa_{s_j}\}_{j=1}^{n_4} \tag{27}$$

interlace

$$\gamma_1 < \eta_1 < \gamma_2 < \eta_2 < \ldots < \eta_{n-1} < \gamma_n < \nu_n := \eta_n. \tag{28}$$

Then there exists a unique couple of sequences of positive numbers $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$, such that $\sum_{k=0}^n l_k = l$, and

1) $\{\nu_n\} \cup \{\nu_{k_j}\}_{j=1}^{n_1}$ *are the eigenvalues of Dirichlet-Dirichlet problem (12)–(14) with the corresponding indices,*

2) $\{\kappa_{s_j}\}_{j=1}^{n_4}$ *are eigenvalues of Neumann-Dirichlet problem (12) (*$k \in \{2, 3, \dots, n\}$*), (14), (21),*

3) $\{\mu_{r_j}\}_{j=1}^{n_3}$ *are eigenvalues of Dirichlet-Neumann problem (12), (13), (18).*

4) $\{\zeta_{p_j}\}_{j=1}^{n_2}$ *are eigenvalues of Neumann-Neumann problem (12) (*$k \in \{2, 3, \dots, n\}$*), (18), (21).*

Proof. Due to (28) and Theorem 2 we conclude that there exists a unique couple of sequences of positive numbers $\{\widetilde{m}_k\}_{k=1}^n$ and $\{\widetilde{l}_k\}_{k=0}^n$ which together with the given l generate problem (12)–(14) with the spectrum $\{\eta_k\}_{k=1}^n$ and problem (12), (13), (21) with the spectrum $\{\gamma_k\}_{k=1}^n$. The corresponding Dirichlet-Neumann characteristic polynomial is

$$\widetilde{R}_{2n-1}(\lambda) = \frac{1}{\widetilde{l}_0} \prod_{k=1}^n \left(1 - \frac{\lambda}{\gamma_k}\right) \tag{29}$$

while the corresponding Dirichlet-Dirichlet characteristic polynomial is

$$\widetilde{R}_{2n}(\lambda) = \frac{l}{\widetilde{l}_0} \prod_{k=1}^n \left(1 - \frac{\lambda}{\eta_k}\right). \tag{30}$$

The corresponding Neumann-Dirichlet and Neumann-Neumann characteristic polynomials $\widetilde{Q}_{2n}(\lambda)$, $\widetilde{Q}_{2n-1}(\lambda)$, respectively, can be found either solving the direct problems (12), (18), (14) and (12), (12), (18) or using the Lagrange identity

$$\widetilde{R}_{2n-1}(\lambda)\widetilde{Q}_{2n}(\lambda) - \widetilde{R}_{2n}(\lambda)\widetilde{Q}_{2n-1}(\lambda) = \frac{1}{\widetilde{l}_0}. \tag{31}$$

The latter method lies in solving interpolation problems. We choose $\lambda = \zeta_k$, $k \in \{1, \dots, n\}$ as the nodes of interpolation and find the values of $\tilde{Q}_{2n}(z)$ at the nodes using (31)

$$\widetilde{Q}_{2n}(\eta_k) = \frac{1}{\widetilde{l}_0 \widetilde{R}_{2n-1}(\eta_k)}, \tag{32}$$

which is well defined because $\widetilde{R}_{2n-1}(\eta_k) \neq 0$ due to (28). We choose $\eta_0 = 0$ as one more node of interpolation and find the corresponding value

$$\widetilde{Q}_{2n}(0) = 1 \tag{33}$$

using recurrence relations analogue to (22)–(23).

Thus, using these data we construct the Lagrange interpolating polynomial

$$\widetilde{Q}_{2n}(\lambda) = \sum_{k=1}^n \frac{\lambda}{\eta_k \widetilde{l}_0 \widetilde{R}_{2n-1}(\eta_k)} \prod_{j=1, j\neq k}^n \frac{\lambda - \eta_j}{\widetilde{\nu}_k - \eta_j} + (-1)^n \prod_{k=1}^n \frac{\lambda - \eta_k}{\eta_k}. \tag{34}$$

In the same way, identity (31) for $\lambda = \gamma_k$, $k \in \{1, \dots, n\}$ gives

$$\widetilde{Q}_{2n-1}(\gamma_k) = -\frac{1}{\widetilde{l}_0 \widetilde{R}_{2n}(\gamma_k)}. \tag{35}$$

We choose $\gamma_0 =: 0$ as one more node of interpolation because the recurrence relations show that

$$\widetilde{Q}_{2n-1}(0) = 0. \tag{36}$$

Therefore, the Lagrange interpolating polynomial is

$$\widetilde{Q}_{2n-1}(\lambda) = -\sum_{k=1}^{n} \frac{\lambda}{\gamma_k \widetilde{l}_0 \widetilde{R}_{2n}(\gamma_k)} \prod_{j=1, j\neq k}^{n} \frac{\lambda - \gamma_j}{\gamma_k - \gamma_j}. \tag{37}$$

Thus, we can find zeros $\{\widetilde{\kappa}_k\}_{k=1}^n$ of the polynomial $\widetilde{Q}_n(z)$ and zeros $\{\widetilde{\zeta}_k\}_{k=0}^{n-1}$ of $\widetilde{Q}_{2n-1}(\lambda)$. The sequence $\{\widetilde{\kappa}_k\}_{k=1}^n$ is the spectrum of Neumann-Dirichlet problem (12), (21), (14) generated by $\{\widetilde{m}_k\}_{k=1}^n$, $\{\widetilde{l}_k\}_{k=0}^n$ while the sequence $\{\widetilde{\zeta}_k\}_{k=0}^{n-1}$ is the spectrum of Neumann-Neumann problem (12), (21), (18) generated by $\{\widetilde{m}_k\}_{k=1}^n$, $\{\widetilde{l}_k\}_{k=0}^n$, therefore they interlace

$$\max\{\gamma_k, \widetilde{\kappa}_k\} < \min\{\eta_k, \widetilde{\zeta}_k\} \leq \max\{\eta_k, \widetilde{\zeta}_k\} < \min\{\gamma_{k+1}, \widetilde{\kappa}_{k+1}\} \leq \max\{\gamma_{k+1}, \widetilde{\kappa}_{k+1}\} < \ldots. \tag{38}$$

Using the notation

$$\mu_k =: \begin{cases} \mu_{r_j}, & k \in \{r_j\}_{j=1}^{n_3}, \\ \widetilde{\kappa}_{s_j}, & k \in \{s_j\}_{j=1}^{n_4}. \end{cases} \tag{39}$$

we compose the sequence

$$\{\mu_k\}_{k=1}^n = \{\mu_{r_j}\}_{j=1}^{n_3} \bigcup \{\widetilde{\kappa}_{s_j}\}_{j=1}^{n_4} \tag{40}$$

and using

$$\nu_k =: \begin{cases} \nu_{k_j}, & k \in \{k_j\}_{j=1}^{n_1}, \\ \widetilde{\zeta}_{p_j}, & k \in \{p_j\}_{j=1}^{n_2}, \\ \nu_n, \end{cases} \tag{41}$$

we choose

$$\{\nu_k\}_{k=1}^n = \{\nu_{k_j}\}_{j=1}^{n_1} \bigcup \{\widetilde{\zeta}_{p_j}\}_{j=1}^{n_2} \bigcup \nu_n. \tag{42}$$

According to (38) the sequences $\{\mu_k\}_{k=1}^n$ and $\{\nu_k\}_{k=1}^n$ satisfy (20) and by Theorem 2 there exists a couple of sequences $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$ of positive numbers which satisfy $\sum_{k=0}^n l_k = l$ and generate problem (12)–(14) with the spectrum $\{\nu_k\}_{k=1}^n$ and problem (12), (13), (18) with the spectrum $\{\mu_k\}_{k=1}^n$.

Since $\{\nu_n\} \cup \{\nu_{k_j}\}_{j=1}^{n_1} \subset \{\nu_k\}_{k=1}^n$ and $\{\mu_{r_j}\}_{j=1}^{n_3} \subset \{\mu_k\}_{k=1}^n$, statements 1) and 3) of Theorem 3 are true.

To prove 2) and 4) of Theorem 3 let us consider Lagrange identity (31) for the problems generated by $\{\widetilde{m}_k\}_{k=1}^n$ and $\{\widetilde{l}\}_{k=0}^n$. Using (29), (30) and the corresponding formulae

$$\widetilde{Q}_{2n}(\lambda) = \prod_{k=1}^{n} \left(1 - \frac{\lambda}{\widetilde{\kappa}_k}\right), \quad \widetilde{Q}_{2n-1}(\lambda) = D\lambda \prod_{k=1}^{n-1} \left(1 - \frac{\lambda}{\widetilde{\zeta}_k}\right)$$

where $D = \sum_{k=1}^n m_k$ we obtain

$$\prod_{k=1}^{n} \left(1 - \frac{\lambda}{\gamma_k}\right) \prod_{k=1}^{n} \left(1 - \frac{\lambda}{\widetilde{\kappa}_k}\right) - l \prod_{k=1}^{n} \left(1 - \frac{\lambda}{\eta_k}\right) D\lambda \prod_{k=1}^{n-1} \left(1 - \frac{\lambda}{\widetilde{\zeta}_k}\right) = 1.$$

Using (26), (27), (39), (41) we obtain

$$\prod_{k=1}^{n}\left(1-\frac{\lambda}{\mu_k}\right)\prod_{j=1}^{n_4}\left(1-\frac{\lambda}{\kappa_{s_j}}\right)\prod_{j=1}^{n_3}\left(1-\frac{\lambda}{\widetilde{\kappa}_{r_j}}\right)-$$
$$-lD\prod_{k=1}^{n}\left(1-\frac{\lambda}{\nu_k}\right)\lambda\prod_{j=1}^{n_1}\left(1-\frac{\lambda}{\zeta_{k_j}}\right)\prod_{j=1}^{n_2}\left(1-\frac{\lambda}{\widetilde{\zeta}_{p_j}}\right)=1.$$

According to Theorem 1 the identity

$$\prod_{k=1}^{n}\left(1-\frac{\lambda}{\mu_k}\right)X-\prod_{k=1}^{n}\left(1-\frac{\lambda}{\nu_k}\right)Y=1$$

has a unique solution (X, Y). This solution is

$$X=\prod_{j=1}^{n_4}\left(1-\frac{\lambda}{\kappa_{s_j}}\right)\prod_{j=1}^{n_3}\left(1-\frac{\lambda}{\widetilde{\kappa}_{r_j}}\right),\quad Y=lD\lambda\prod_{j=1}^{n_1}\left(1-\frac{\lambda}{\zeta_{k_j}}\right)\prod_{j=1}^{n_2}\left(1-\frac{\lambda}{\widetilde{\zeta}_{p_j}}\right).$$

On the other hand, Lagrange identity for the problems generated by $\{m_k\}_{k=1}^{n}$, $\{l_k\}_{k=0}^{n}$ shows that the zeros of X are nothing but the eigenvalues of Neumann-Dirichlet problem generated by $\{m_k\}_{k=1}^{n}$ and $\{l_k\}_{k=0}^{n}$, while the zeros of Y are the zeros of Neumann-Neumann problem generated by the same masses and the same subintervals. Thus we have proved that $\{\kappa_{s_j}\}_{j=1}^{n_4}$ are eigenvalues of the Neumann-Dirichlet problem and $\{\zeta_{s_j}\}_{j=1}^{n_1}$ are eigenvalues of the Neumann-Neumann problem. □

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