JOURNAL OF MATHEMATICAL PHYSICS VOLUME 42, NUMBER 1 JANUARY 2001

Inverse scattering for a Schrödinger equation with energy dependent potential

Cornelis van der Mee[a)]
Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

Vjacheslav Pivovarchik[b)]
Odessa State University of Civil Engineering and Architecture, Didrichson Ul. 4, 270029 Odessa, Ukraine

(Received 15 June 2000; accepted for publication 27 September 2000)

In this article the inverse scattering problem of reconstructing the energy dependent potential $i\sqrt{E^2-m^2}\,P(x)+Q(x)$ of a Schrödinger equation on the line from its reflection coefficients and bound state data (i.e., poles of the transmission coefficients and associated norming constants) is solved using the Marchenko integral equation approach.
[DOI: 10.1063/1.1326921]

I. INTRODUCTION

In this article we study the inverse scattering problem for the generalized 1-D Schrödinger equation,

$$\psi''(k,x)+[k^2+m^2]\psi(k,x)=[ikP(x)+Q(x)]\psi(k,x), \quad x\in\mathbb{R}, \tag{1.1}$$

where the prime denotes the derivative with respect to the spatial coordinate x, k is the wavenumber, m is a positive mass parameter, $P(x)$ describes the energy absorption or generation, and $Q(x)$ represents the restoring force density. The quantity $E=\sqrt{k^2+m^2}$ stands for the energy.

Letting $\mathbb{C}^+$ and $\mathbb{C}^-$ stand for the open upper and lower complex half-planes and defining the regions $\Omega^+=\mathbb{C}^+\setminus i[0,m]$ and $\Omega^-=\mathbb{C}^-\setminus i[-m,0]$, for a suitable choice of the square root one can use the mapping $E=\sqrt{k^2+m^2}$ to transform either of the regions $\Omega^\pm$ conformally and bijectively into either of the regions $\mathbb{C}^\pm$, thus yielding four transformations. Using the inverse transformation $k(E)=\sqrt{E^2-m^2}$ we obtain the two-fold Riemann surface with branch cuts along the real line from m to $+\infty$ and from $-m$ to $-\infty$. As we are interested primarily in the domain $E\in\mathbb{C}^+\cup\mathbb{R}$, it is natural to define $k(E)=\sqrt{E^2-m^2}$ as a single-valued continuous function of $E\in\mathbb{C}^+\cup\mathbb{R}$ with $(k(E)/E)>0$ for $E\in\mathbb{R}\setminus[-m,m]$, so that $\operatorname{Im}k(E)>0$ for $E\in(-m,m)$. We then write (1.1) in the equivalent form

$$\psi^{\pm\prime\prime}(E,x)+E^2\psi^\pm(E,x)=[\pm i\,k(E)P(x)+Q(x)]\psi^\pm(E,x), \tag{1.2}$$

where $x\in\mathbb{R}$ and $E\in\overline{\mathbb{C}^+}$.

Let us define the Jost solutions $f_l^\pm(E,x)$ and $f_r^\pm(E,x)$ as the solutions of (1.2) with the $\pm$ sign in the first term of the right-hand side that satisfy the boundary conditions

$$\begin{aligned} f_l^\pm(E,x)&=e^{iEx}+o(1), \quad x\to+\infty,\\ f_r^\pm(E,x)&=e^{-iEx}+o(1), \quad x\to-\infty. \end{aligned} \tag{1.3}$$

[a)]Electronic mail: cornelis@unica.it Research supported in part by INDAM-GNIM and MURST.
[b)]Electronic mail: v.pivovarchik@paco.net, vnp.@dtp.odessa.ua Research supported in part by INDAM-GNFM.

In terms of the Jost solutions, the scattering coefficients $a_l^{\pm}(E)$, $a_r^{\pm}(E)$, $b_l^{\pm}(E)$, and $b_r^{\pm}(E)$ are defined by

$$\begin{aligned} f_l^{\pm}(E,x)&=a_l^{\pm}(E)e^{iEx}+b_l^{\pm}(E)e^{-iEx}+o(1), \quad x\to-\infty,\\ f_r^{\pm}(E,x)&=a_r^{\pm}(E)e^{-iEx}+b_r^{\pm}(E)e^{iEx}+o(1), \quad x\to+\infty. \end{aligned} \tag{1.4}$$

In this article a fundamental role is played by the transformation

$$g_1(E)=\frac{g^+(E)+g^-(E)}{2}, \quad g_2(E)=\frac{g^+(E)-g^-(E)}{2ik(E)}, \tag{1.5}$$

between pairs of functions of E. This transformation allows one to convert the pair of uncoupled differential equations (1.2) into the coupled system of differential equations,

$$\begin{bmatrix}\psi_1''(E,x)\\ \psi_2''(E,x)\end{bmatrix}+E^2\begin{bmatrix}\psi_1(E,x)\\ \psi_2(E,x)\end{bmatrix}=\Xi(E,x)\begin{bmatrix}\psi_1(E,x)\\ \psi_2(E,x)\end{bmatrix}, \tag{1.6}$$

where

$$\Xi(E,x)=\begin{bmatrix}Q(x) & -k(E)^2P(x)\\ P(x) & Q(x)\end{bmatrix} \tag{1.7}$$

and $k(E)^2=E^2-m^2$. Transforming the Jost solutions as in (1.5), we obtain from (1.3) and (1.4),

$$\begin{aligned} f_{l1}(E,x)&=e^{iEx}+o(1), \quad f_{l2}(E,x)=o(1), \quad x\to+\infty,\\ f_{r1}(E,x)&=e^{-iEx}+o(1), \quad f_{r2}(E,x)=o(1), \quad x\to-\infty, \end{aligned} \tag{1.8}$$

$$\begin{aligned} f_{ls}(E,x)&=a_{ls}(E)e^{iEx}+b_{ls}(E)e^{-iEx}+o(1), \quad x\to-\infty,\\ f_{rs}(E,x)&=a_{rs}(E)e^{-iEx}+b_{rs}(E)e^{iEx}+o(1), \quad x\to+\infty, \end{aligned} \tag{1.9}$$

where $s=1,2$.

The direct and inverse scattering problems for Schrödinger equations of the type (1.1) have been studied extensively. Jaulent and Jean[1–3] studied (1.1) with $m=0$, imaginary $P(x)$ and real $Q(x)$, both on the half-line and on the full line (problems leading to unitary scattering data), and established the unique solvability of their Marchenko equations. Jaulent[4] derived Marchenko integral equations leading to the solution of the inverse problem for (1.1) with $m=0$ and real potentials $P(x)$ and $Q(x)$. Sattinger and Szmigielski[5] studied (1.1) with $m=0$, imaginary $P(x)$ and real $Q(x)$ and applied the results to solve a nonlinear evolution equation. Aktosun *et al.*[6,7] studied in detail the direct and inverse scattering problems for (1.2) for $m=0$, obtained many results on the discrete eigenvalues, and gave sufficient conditions for the unique solvability of the Marchenko equations.

The more interesting case where $m>0$, was taken up by Kaup[8,9] in connection with a nonlinear evolution equation (a long-wave water equation resembling the Boussinesq equation). In Ref. 9 a pair of coupled Marchenko integral equations was given to solve the inverse scattering problem. Under the assumption that $\int_{-\infty}^{\infty}dx\, P(x)=0$, Sattinger and Szmigielski[10] considered the direct and inverse problems for (1.1) with $m=1$ and C^{∞} potentials and applied their results to a nonlinear evolution equation. Equation (1.1), with k^2+m^2 and $ikP(x)$ replaced by k^2-m^2 and $kP(x)$, respectively, for real potentials $P(x)$ and $Q(x)$, is the 1-D Klein–Gordon equation. For this equation and on the half-line, Corinaldesi,[11] Degasperis,[12] and Weiss and Scharf[13] studied the inverse scattering problem and Pivovarchik[14] studied the number of bound states.

When $P,Q\in L^1(\mathbb{R})$, the Schrödinger operators in (1.2) have very different properties depending on whether $m=0$ or $m>0$, since for $m=0$ their essential spectrum is the set of $k\in\mathbb{R}$ whereas for $m>0$ it is the set of $k\in\mathbb{R}\cup i[-m,m]$. Moreover, as observed in Refs. 8 and 9, the $m>0$ equation is important for solving a certain system of nonlinear evolution equations by the inverse scattering transform, whereas no such connection is apparent for $m=0$.

In this article we analyze the inverse scattering problem for (1.2) by the Marchenko method. Essentially, although most of the scattering solutions and scattering coefficients are defined as in Refs. 6 and 7 where the $m=0$ case was treated, we differ from these papers in one important aspect: We also define the scattering solutions and scattering coefficients as if (1.6) were the equation of interest rather than (1.2). The Riemann–Hilbert problem relating the usual Faddeev solutions, as studied since the seminal papers by Faddeev[15] and Deift and Trubowitz,[16] and the Marchenko integral equations obtained by Fourier transformation are derived for quantities that are primarily connected with (1.6). The relationships between the two approaches are explained in detail. The advantage of the new approach lies in the behavior as $E\to\pm m$. In this case, (1.2) approaches two copies of the 1-D Schrödinger equation on the line with real potential $Q(x)$, whereas (1.6) tends to a nonselfadjoint matrix Schrödinger equation that also involves $P(x)$. In principle, this new approach could also have been applied to the case $m=0$, a possibility not observed before. It might then be comparatively easy to study the behavior of the solutions of (1.2) as $m\to 0^+$.

Let us discuss briefly some of the differences between Ref. 10 and the present paper. In Refs. 9 and 10 the E and k variables are transformed into the complex z variable by the conformal mapping $z=E+k=m^2/(E-k)$, where $m=1$ in Ref. 10. The complex z-plane is then divided into the regions $\mathcal{U}_+=\{z\in\mathbb{C}:|z|>m \text{ and } \operatorname{Im} z>0\}\cup\{z\in C:|z|<m \text{ and } \operatorname{Im} z<0\}$ and $\mathcal{U}_-=\{z\in\mathbb{C}:|z|>m$ and $\operatorname{Im} z<0\}\cup\{z\in C:|z|<m \text{ and } \operatorname{Im} z>0\}$, separated by $\Sigma=\{z\in\mathbb{C}:|z|=m\}\cup(\mathbb{R}\setminus\{0\})$. The inverse scattering problem is then posed as a vector Riemann–Hilbert problem on the curve Σ that relates vector functions analytic in $\mathcal{U}_-$ to vector functions analytic in $\mathcal{U}_+$. The unfamiliarity of the curve Σ, however, makes it hard to replace these Riemann–Hilbert problems by equivalent integral equations. For this reason we have decided not to use the z variable.

Let us now discuss the contents of this article. In Sec. II we introduce and study the scattering solutions and their asymptotic properties as $|E|\to\infty$. We also derive the continuity of the scattering solutions for (1.6) as $E\to\pm m$. In Sec. III we introduce and study the scattering coefficients and their asymptotics as $|E|\to\infty$. Their behavior as $E\to\pm m$ is also obtained. Their asymptotics as $E\to 0$ is found using the recent results in Ref. 17. It follows in particular that the scattering matrix is unitary if $E\in[-m,m]$, something that can also be derived from results in Ref. 10, and has certain contractivity and expansivity properties if $E\in\mathbb{R}\setminus[-m,m]$ and $P(x)$ does not change sign. In Sec. IV an idea by Weiss and Scharf[13] is employed to derive Marchenko integral equations for (1.6), both in the absence and in the presence of (finitely many) discrete eigenvalues. Any solution of one of the two coupled systems of two Marchenko integral equations allows one to uniquely determine the potentials $P(x)$ and $Q(x)$, provided the second one of the pair of functions being a solution has its values in $(-1,1)$. In Sec. V we relate, as in Ref. 7, the unique solvability of either of the systems of Marchenko equations to the existence of a canonical Wiener–Hopf factorization of a 2×2 matrix function on the line.

II. JOST SOLUTIONS AND FADDEEV FUNCTIONS

In this section we introduce various scattering solutions for (1.2) and (1.6) and study their symmetry and asymptotic properties.

A. Analyticity and symmetry properties

Let $P,Q\in L^1(\mathbb{R})$. Then the Jost solutions $f_l^{\pm}(E,x)$ and $f_r^{\pm}(E,x)$ satisfy the integral equations

$$f_l^{\pm}(E,x)=e^{iEx}+\frac{1}{E}\int_x^\infty dy\,\sin\{E(y-x)\}[\pm i\,k(E)P(y)+Q(y)]f_l^{\pm}(E,x); \tag{2.1}$$

$$f_r^{\pm}(E,x)=e^{-iEx}+\frac{1}{E}\int_{-\infty}^{x}dy\,\sin\{E(x-y)\}[\pm i\,k(E)P(y)+Q(y)]f_r^{\pm}(E,x). \tag{2.2}$$

Using the 2×2 matrix $\Xi(E,x)$ introduced in (1.7), these integral equations are easily transformed into the pairs of coupled integral equations,

$$\begin{bmatrix} f_{l1}(E,x) \\ f_{l2}(E,x) \end{bmatrix}=\begin{bmatrix} \cos(Ex) \\ \dfrac{\sin(Ex)}{k(E)} \end{bmatrix}+\int_x^{\infty}dy\,\frac{\sin\{E(y-x)\}}{E}\,\Xi(E,y)\begin{bmatrix} f_{l1}(E,y) \\ f_{l2}(E,y) \end{bmatrix};$$

$$\begin{bmatrix} f_{r1}(E,x) \\ f_{r2}(E,x) \end{bmatrix}=\begin{bmatrix} \cos(Ex) \\ \dfrac{\sin(Ex)}{k(E)} \end{bmatrix}+\int_{-\infty}^{x}dy\,\frac{\sin\{E(x-y)\}}{E}\,\Xi(E,y)\begin{bmatrix} f_{r1}(E,y) \\ f_{r2}(E,y) \end{bmatrix}.$$

Defining the Faddeev functions $m_l^{\pm}(E,x)$ and $m_r^{\pm}(E,x)$ by

$$m_l^{\pm}(E,x)=e^{-iEx}f_l^{\pm}(E,x),\quad m_r^{\pm}(E,x)=e^{iEx}f_r^{\pm}(E,x), \tag{2.3}$$

we get from (2.1) and (2.2) the Volterra integral equations,

$$m_l^{\pm}(E,x)=1+\int_x^{\infty}dy\,\frac{e^{2iE(y-x)}-1}{2iE}[\pm i\,k(E)P(y)+Q(y)]m_l^{\pm}(E,y); \tag{2.4}$$

$$m_r^{\pm}(E,x)=1+\int_{-\infty}^{x}dy\,\frac{e^{2iE(x-y)}-1}{2iE}[\pm i\,k(E)P(y)+Q(y)]m_r^{\pm}(E,y). \tag{2.5}$$

Using (1.5), these are transformed into the pairs of coupled integral equations:

$$\begin{bmatrix} m_{l1}(E,x) \\ m_{l2}(E,x) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\int_x^{\infty}dy\,\frac{e^{2iE(y-x)}-1}{2iE}\,\Xi(E,y)\begin{bmatrix} m_{l1}(E,y) \\ m_{l2}(E,y) \end{bmatrix}; \tag{2.6}$$

$$\begin{bmatrix} m_{r1}(E,x) \\ m_{r2}(E,x) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\int_{-\infty}^{x}dy\,\frac{e^{2iE(x-y)}-1}{2iE}\,\Xi(E,y)\begin{bmatrix} m_{r1}(E,y) \\ m_{r2}(E,y) \end{bmatrix}. \tag{2.7}$$

By differentiation with respect to x we obtain

$$\begin{bmatrix} m_{l1}'(E,x) \\ m_{l2}'(E,x) \end{bmatrix}=-\int_x^{\infty}dy\,e^{2iE(y-x)}\,\Xi(E,y)\begin{bmatrix} m_{l1}(E,y) \\ m_{l2}(E,y) \end{bmatrix}; \tag{2.8}$$

$$\begin{bmatrix} m_{r1}'(E,x) \\ m_{r2}'(E,x) \end{bmatrix}=\int_{-\infty}^{x}dy\,e^{2iE(x-y)}\,\Xi(E,y)\begin{bmatrix} m_{r1}(E,y) \\ m_{r2}(E,y) \end{bmatrix}. \tag{2.9}$$

In the next theorem we state the analyticity and continuity properties of $m_{ls}(E,x)$, $m_{rs}(E,x)$, $f_{ls}(E,x)$, $f_{rs}(E,s)$ and their derivatives $(s=1,2)$. Such results will then also hold for $m_l^{\pm}(E,x)$, $m_r^{\pm}(E,x)$, $f_l^{\pm}(E,x)$, $f_r^{\pm}(E,x)$ and their derivatives.

Theorem 2.1: *Assume $P,Q\in L^1(\mathbb{R})$. Then the following is true.*

(1) For $x\in\mathbb{R}$ and $s=1,2$, the functions $m_{ls}(E,x)$, $m_{rs}(E,x)$, $m_{ls}'(E,x)$ and $m_{rs}'(E,x)$ are analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\backslash\{0\}$. Consequently, for each $x\in\mathbb{R}$ and $s=1,2$ the transformed Jost solutions $f_{ls}(E,x)$ and $f_{rs}(E,x)$ and their derivatives $f_{ls}'(E,x)$ and $f_{rs}'(E,x)$ are analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\backslash\{0\}$.

(2) If $P,Q\in L_1^1(\mathbb{R})$, the continuity of the functions in (i) *extends to $\overline{\mathbb{C}^+}$.*

Proof: Let $E \in \overline{\mathbb{C}^+}$. For large $|E|$ it is expedient to iterate the four integral equations (2.4) and (2.5) for $m_l^{\pm}(E,x)$ and $m_r^{\pm}(E,x)$, while it is more convenient to iterate the two systems (2.6) and (2.7) as $E \to \pm m$.

First, using the estimate

$$\sqrt{||E|^2-m^2|} \leqslant |k(E)| \leqslant \sqrt{|E|^2+m^2},$$

one obtains the auxiliary upper bounds:

$$\left|\frac{e^{2iE|y-x|}-1}{2iE}[\pm i\,k(E)P(y)+Q(y)]\right| \leqslant \begin{cases} 2|P(y)|+\dfrac{1}{m}|Q(y)|, & |E|\geqslant m, \\ (m\sqrt{2}\,|P(y)|+|Q(y)|)/|E|, & |E|\leqslant m, \\ |y-x|[m\sqrt{2}\,|P(y)|+|Q(y)|], & |E|\leqslant m. \end{cases}$$

In analogy with Refs. 15 and 16 we obtain the estimates

$$|m_l^{\pm}(E,x)| \leqslant \begin{cases} \exp\left(2\|P\|_1+\dfrac{1}{m}\|Q\|_1\right), & |E|\geqslant m, \\ \exp((m\sqrt{2}\,\|P\|_1+\|Q\|_1)/|E|), & |E|\leqslant m, \\ [1+\max(0,-x)]\exp(m\sqrt{2}\,\|P\|_{1,1}+\|Q\|_{1,1}), & |E|\leqslant m, \end{cases}$$

and hence

$$\max(|m_l^{\pm}(E,x)|,|m_r^{\pm}(E,x))\leqslant c_2;$$

$$\max(|m_l^{\pm\prime}(E,x)|,|m_r^{\pm\prime}(E,x)|)\leqslant c_1c_2(\|P\|_1+\|Q\|_1)[1+|E|],$$

where $c_2=e^{c_1(\|P\|_1+\|Q\|_1)/\min(1,|E|)}$ and $c_1=\max(2,m\sqrt{2},1/m)$, as well as

$$|m_l^{\pm}(E,x)|\leqslant[1+\max(0,-x)]e^{c_1(\|P\|_{1,1}+\|Q\|_{1,1})}; \tag{2.10}$$

$$|m_l^{\pm\prime}(E,x)|\leqslant c_1c_3(\|P\|_1+\|Q\|_1)[1+|E|][1+\max(0,-x)], \tag{2.11}$$

where $c_3=e^{c_1(\|P\|_{1,1}+\|Q\|_{1,1})}$. The proof for $m_r^{\pm}(E,x)$ and $m_r^{\pm\prime}(E,x)$, where (2.10) and (2.11) hold with $\max(0,-x)$ replaced by $\max(0,x)$, is similar.

Next, the derivation of the analyticity of $m_{ls}(E,x)$ and $m_{rs}(E,x)$ and their derivatives in a neighborhood of $\pm m$ in $\overline{\mathbb{C}^+}$ for $s=1,2$ is analogous. Here one employs the following estimate for the Euclidean norm of the matrix $\Xi(E,x)$:

$$\|\Xi(E,x)\|\leqslant 2|Q(x)|+(1+|k(E)|^2)|P(x)|,$$

which completes the proof. □

When $P,Q\in L_1^1(\mathbb{R})$, we find as $E\to 0$,

$$m_l^{\pm}(0,x)=1+\int_x^{\infty}dy\,(y-x)[Q(y)\mp mP(y)]m_l^{\pm}(0,y); \tag{2.12}$$

$$m_r^{\pm}(0,x)=1+\int_{-\infty}^{x}dy\,(x-y)[Q(y)\mp mP(y)]m_r^{\pm}(0,y). \tag{2.13}$$

Then (2.12) and (2.13) are the integral equations for the zero energy Jost functions of the usual 1-D Schrödinger equation with potential $Q(x)\mp mP(x)$. We will call $Q\mp mP$ an *exceptional* potential (for the usual Schrödinger equation) if there exists a nonzero (real) constant $\gamma^{\pm}$ such that

$$\gamma^{\pm}=\frac{m_l^{\pm}(0,x)}{m_r^{\pm}(0,x)}=\frac{f_l^{\pm}(0,x)}{f_r^{\pm}(0,x)}. \tag{2.14}$$

Otherwise $Q\mp mP$ is called a *generic* potential. Obviously, (2.12) and (2.13) can be transformed into the pairs of coupled integral equations

$$\begin{bmatrix} m_{l1}(0,x) \\ m_{l2}(0,x) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\int_x^{\infty} dy\,(y-x)\begin{bmatrix} Q(y) & m^2P(y) \\ P(y) & Q(y) \end{bmatrix}\begin{bmatrix} m_{l1}(0,y) \\ m_{l2}(0,y) \end{bmatrix};$$

$$\begin{bmatrix} m_{r1}(0,x) \\ m_{r2}(0,x) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\int_{-\infty}^{x} dy\,(x-y)\begin{bmatrix} Q(y) & m^2P(y) \\ P(y) & Q(y) \end{bmatrix}\begin{bmatrix} m_{r1}(0,y) \\ m_{r2}(0,y) \end{bmatrix}.$$

The complicated conjugation symmetry properties of $k(E)$ make it hazardous to state conjugation symmetry properties for $f_l^{\pm}(E,x)$, $f_r^{\pm}(E,x)$, $m_l^{\pm}(E,x)$, and $m_r^{\pm}(E,x)$ directly. However, since $k(-\overline{E})^2=\overline{k(E)^2}$, we immediately have for $s=1,2$,

$$\overline{f_{ls}(-\overline{E},x)}=f_{ls}(E,x),\quad \overline{f_{rs}(-\overline{E},x)}=f_{rs}(E,x), \tag{2.15}$$

and similarly for $m_{ls}(E,x)$ and $m_{rs}(E,x)$ where $s=1,2$. Now note that $(k(E)/E)>0$ for $E\in\mathbb{R}\backslash[-m,m]$ and $k(E)$ is positive imaginary for $E\in(-m,m)$. Thus $\overline{k(-\overline{E})}=-k(E)$ for $E\in\overline{\mathbb{C}^+}$. Using the identities $f_l^{\pm}(E,x)=f_{l1}(E,x)\pm ik(E)f_{l2}(E,x)$ and similarly for $f_r^{\pm}(E,x)$, we obtain

$$\overline{f_l^{\pm}(-\overline{E},x)}=f_l^{\pm}(E,x),\quad \overline{f_r^{\pm}(-\overline{E},x)}=f_r^{\pm}(E,x). \tag{2.16}$$

Similar relations hold for $m_l^{\pm}(E,x)$ and $m_r^{\pm}(E,x)$.

B. Large-*E* asymptotics

To study the large-E asymptotics of the Jost solutions, we define

$$\eta_l^{\pm}(E,x)=e^{\pm\zeta(x)}m_l^{\pm}(E,x)=e^{-iEx\pm\zeta(x)}f_l^{\pm}(E,x); \tag{2.17}$$

$$\eta_r^{\pm}(E,x)=e^{\pm p\mp\zeta(x)}m_l^{\pm}(E,x)=e^{iEx\pm p\mp\zeta(x)}f_r^{\pm}(E,x), \tag{2.18}$$

where

$$\zeta(x)=\frac{1}{2}\int_x^{\infty}dz\,P(z),\quad p=\frac{1}{2}\int_{-\infty}^{\infty}dz\,P(z). \tag{2.19}$$

Theorem 2.2: *Let* $P,Q\in L^1(\mathbb{R})$. *Then the following statements are true.*

(i) For each $x\in\mathbb{R}$, *the functions* $\eta_l^{\pm}(E,x)$ *and* $\eta_r^{\pm}(E,x)$ *are analytic in* $\mathbb{C}^+$, *are continuous in* $\overline{\mathbb{C}^+}\backslash\{0\}$, *and we have for some constant C not depending on k and x,*

$$|\eta_l^{\pm}(E,x)|\leq Ce^{C/|E|},\quad |\eta_r^{\pm}(E,x)|\leq Ce^{C/|E|},\quad E\in\overline{\mathbb{C}^+}\backslash\{0\}. \tag{2.20}$$

Further, as $|E|\to\infty$ *in* $\overline{\mathbb{C}^+}$ *we have*

$$\eta_l^{\pm}(E,x)=1+o(1),\quad \eta_r^{\pm}(E,x)=1+o(1); \tag{2.21}$$

$$\eta_l^{\pm\prime}(E,x)=o(E),\quad \eta_r^{\pm\prime}(E,x)=o(E). \tag{2.22}$$

(ii) If $P,Q \in L^1_1(\mathbb{R})$, the continuity of the functions in (i) *extends to $\overline{\mathbb{C}^+}$. Moreover, for $E \in \overline{\mathbb{C}^+}$ we have*

$$|\eta_l^{\pm}(E,x)| \leq C[1+\max(0,-x)], \quad |\eta_r^{\pm}(E,x)| \leq C[1+\max(0,x)].$$

Proof: Letting $z(E,x)=\eta_l^{\pm}(E,x)-1$, we obtain

$$z(E,x)=z_0(E,x)+\int_x^\infty dy\,\frac{e^{2iE(y-x)}-1}{2iE}\,e^{\pm[\zeta(x)-\zeta(y)]}[\pm i\,k(E)P(y)+Q(y)]z(E,y), \tag{2.23}$$

where

$$z_0(E,x)=\int_x^\infty dy\,\frac{e^{2iE(y-x)}-1}{2iE}\,e^{\pm[\zeta(x)-\zeta(y)]}Q(y)$$
$$\pm\frac{k(E)}{2E}\int_x^\infty dy\,e^{2iE(y-x)}e^{\pm[\zeta(x)-\zeta(y)]}P(y)+\left(1-\frac{k(E)}{E}\right)[e^{\pm\zeta(x)}-1].$$

Then the Riemann–Lebesgue lemma implies that $\sup_{t\geq x}|z_0(E,t)|$ vanishes as $E\to\pm\infty$. Iterating (2.23) we now see that $z(E,x)$ is uniformly bounded in $\overline{\mathbb{C}^+}$ for $|E|\geq a>0$ for each $x\in\mathbb{R}$ and $a>0$. Using a Phragmen–Lindelöf theorem (cf. Ref. 18) we conclude that $z(E,x)$ vanishes as $E\to\infty$ in $\overline{\mathbb{C}^+}$.

To prove (2.22) we introduce the function

$$\xi_l^{\pm}(E,x)=\frac{1}{iE}m_l^{\pm\prime}(E,x)e^{\pm\zeta(x)}=\frac{1}{2iE}[\pm P(x)\,\eta_l^{\pm}(E,x)+2\,\eta_l^{\pm\prime}(E,x)].$$

From (2.8) and (2.17) we get

$$\xi_l^{\pm}(E,x)=\int_x^\infty dy\,e^{2iE(y-x)}\left[\mp\frac{k(E)}{E}P(y)+\frac{i}{E}Q(y)\right]e^{\pm[\zeta(x)-\zeta(y)]}\eta_l^{\pm}(E,y). \tag{2.24}$$

Thus, using (2.20), we see that the integrand on the right-hand side of (2.24) is bounded by the integrable function $C_a[|P(y)|+|Q(y)|]$, uniformly in $x\in\mathbb{R}$ and $E\in\overline{\mathbb{C}^+}$ for $|E|\geq a>0$ and each $a>0$, where the constant C_a does not depend on x and E. By the Riemann–Lebesgue lemma, we conclude that the right-hand side of (2.24) is $o(1)$ as $E\to\pm\infty$, so that by a Phragmen–Lindelöf theorem (cf. Ref. 18) we see that the left-hand side of (2.24) is $o(1)$ as $E\to\infty$ in $\overline{\mathbb{C}^+}$. Consequently, $\xi_l^{\pm}(E,x)=o(1)$ as $E\to\infty$ in $\overline{\mathbb{C}^+}$, which implies (2.22) for $\eta_l^{\pm\prime}(E,x)$. The proof for $\eta_r^{\pm}(E,x)$ and $\eta_r^{\pm\prime}(E,x)$ is similar. □

To study the inverse scattering problem for (1.2), as in Ref. 7 we strengthen Theorem 2.2 by making additional assumptions on P and Q. In fact, we assume that P is absolutely continuous, and define the two auxiliary potential functions,

$$W^{\pm}(x)=Q(x)\mp\tfrac{1}{2}P'(x)-\tfrac{1}{4}P(x)^2. \tag{2.25}$$

Using (1.2)–(1.4) we obtain for $x\in\mathbb{R}$,

$$\eta_l^{\pm\prime\prime}(E,x)+[2iE\pm P(x)]\eta_l^{\pm\prime}(E,x)=[W^{\pm}(x)\mp i(E-k(E))P(x)]\eta_l^{\pm}(E,x), \tag{2.26}$$

$$\eta_l^{\pm}(E,+\infty)=1, \quad \eta_l^{\pm\prime}(E,+\infty)=0, \tag{2.27}$$

where $W^{\pm}(x)$ is given by (2.25). Multiplying (2.26) by $\mu_l^{\pm}(E,x)=e^{2iEx\mp2\zeta(x)}$, we obtain for $x\in\mathbb{R}$,

$$[\mu_l^{\pm}(E,x)\,\eta_l^{\pm\prime}(E,x)]' = \mu_l^{\pm}(E,x)\,[W^{\pm}(x) \mp i(E-k(E))P(x)]\,\eta_l^{\pm}(E,x). \tag{2.28}$$

Integrating (2.28) and using (2.27) we get

$$\eta_l^{\pm\prime}(E,x) = -\int_x^{\infty} dy\, e^{2iE(y-x)\pm\int_x^y d\hat{z}\, P(\hat{z})}[W^{\pm}(y) \mp i(E-k(E))P(y)]\,\eta_l^{\pm}(E,y). \tag{2.29}$$

Integrating (2.29), using (2.27) once again and changing the order of integration, we find

$$\eta_l^{\pm}(E,x) = 1 + \int_x^{\infty} dy\, G_l^{\pm}(E;x,y)[W^{\pm}(y) \mp i(E-k(E))P(y)]\,\eta_l^{\pm}(E,y), \tag{2.30}$$

where we have defined

$$\begin{aligned} G_l^{\pm}(E;x,y) &= \int_x^y dz\, e^{2iE(y-z)\pm\int_z^y d\hat{z}\, P(\hat{z})} \\ &= \frac{1}{2iE}[e^{2iE(y-x)\pm\int_x^y d\hat{z}\, P(\hat{z})} - 1] \mp \frac{1}{2iE}\int_x^y dz\, P(z)\, e^{2iE(y-z)\pm\int_z^y d\hat{z}\, P(\hat{z})}. \end{aligned} \tag{2.31}$$

Similarly, using (1.2)–(1.4), (2.17)–(2.19), and (2.25) we obtain

$$\eta_r^{\pm\prime\prime}(E,x) - [2iE \pm P(x)]\,\eta_r^{\pm\prime}(E,x) = [W^{\mp}(x) \pm i(E-k(E))P(x)]\,\eta_r^{\pm}(E,x), \tag{2.32}$$

$$\eta_r^{\pm}(E,-\infty) = 1, \quad \eta_r^{\pm\prime}(E,-\infty) = 0. \tag{2.33}$$

Integrating (2.32) twice and using (2.33) we first get

$$\eta_r^{\pm\prime}(E,x) = \int_{-\infty}^{x} dy\, e^{2iE(x-y)\pm\int_y^x d\hat{z}\, P(\hat{z})}[W^{\mp}(y) \pm i(E-k(E))P(y)]\,\eta_r^{\pm}(E,y),$$

and subsequently

$$\eta_r^{\pm}(E,x) = 1 + \int_{-\infty}^{x} dy\, G_r^{\pm}(E;x,y)\,[W^{\mp}(y) \pm i(E-k(E))P(y)]\,\eta_r^{\pm}(E,y), \tag{2.34}$$

where we have defined

$$\begin{aligned} G_r^{\pm}(E;x,y) &= \int_y^x dz\, e^{2iE(z-y)\pm\int_y^z d\hat{z}\, P(\hat{z})} \\ &= \frac{1}{2iE}[e^{2iE(x-y)\pm\int_y^x P(z)dz} - 1] \pm \frac{1}{2iE}\int_y^x dz\, P(z)\, e^{2iE(z-y)\pm\int_y^z d\hat{z}\, P(\hat{z})}. \end{aligned}$$

Let us now employ the integral equations (2.30) and (2.34) to derive asymptotic expressions for $\eta_l^{\pm}(E,x)$ and $\eta_r^{\pm}(E,x)$ as $E \to \infty$ in $\overline{\mathbb{C}^+}$.

Theorem 2.3: *(1) Assume $P \in L^1(\mathbb{R})$ and $Q \in L_1^1(\mathbb{R})$. Then, for each fixed $x \in \mathbb{R}$, the functions $\eta_l^{\pm}(E,x)$ and $\eta_r^{\pm}(E,x)$ are analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}$, and*

$$\eta_l^{\pm}(E,x) = 1 + o(1), \quad \eta_r^{\pm}(E,x) = 1 + o(1), \quad E \to \infty \text{ in } \overline{\mathbb{C}^+}.$$

(2) Assume that $W^+, W^- \in L^1(\mathbb{R})$. Then as $E \to \infty$ in $\overline{\mathbb{C}^+}$ we have

$$\eta_l^{\pm}(E,x) = 1 + O(1/|E|), \quad \eta_r^{\pm}(E,x) = 1 + O(1/|E|). \tag{2.35}$$

(3) If further we assume $P\in L^1_1(\mathbb{R})$, $Q\in L^1_2(\mathbb{R})$ *and* $W^\pm\in L^1_1(\mathbb{R})$, *then*

$$\eta_l^\pm(E,x)=1-\frac{\int_x^\infty dz\,W^\pm(z)}{2iE}+O(1/|E|^2),\quad E\to\infty\ \text{in}\ \overline{\mathbb{C}^+}, \tag{2.36}$$

$$\eta_r^\pm(E,x)=1-\frac{\int_{-\infty}^x dz\,W^\mp(z)}{2iE}+O(1/|E|^2),\quad E\to\infty\ \text{in}\ \overline{\mathbb{C}^+}. \tag{2.37}$$

Proof: We only prove (2.35), (2.36) and (2.37), because the rest of the proof is given in Theorem 3.1 of Ref. 6. Note that (2.31) implies for $y\geqslant x$,

$$|G_l^\pm(E;x,y)|\leqslant\frac{C}{|E|},\quad E\in\overline{\mathbb{C}^+}\setminus\{0\}, \tag{2.38}$$

where $C=\frac{1}{2}(1+(1+||P||_1)e^{||P||_1})$. Thus, iterating (2.30) and using (2.38) we obtain

$$|\eta_l^\pm(E,x)-1|\leqslant\frac{C}{|E|}\left[\int_x^\infty dt\,(|W^\pm(t)|+m|P(t)|)\right]\exp\left(\int_x^\infty dz\,(|W^\pm(z)|+m|P(z)|)\right),$$

where $E\in\overline{\mathbb{C}^+}\setminus\{0\}$ and $|E|\geqslant m$. This implies (2.35) for $\eta_l^\pm(E,x)$ whenever $W^\pm\in L^1(\mathbb{R})$. The proof of (2.35) for $\eta_r^\pm(E,x)$ is obtained in a similar manner. To prove (2.36) we obtain from (2.30),

$$\begin{aligned}\eta_l^\pm(E,x)=1&+\int_x^\infty dy\,G_l^\pm(E;x,y)[W^\pm(y)\mp i(E-k(E))P(y)]\\&+\int_x^\infty dy\,G_l^\pm(E;x,y)[W^\pm(y)\mp i(E-k(E))P(y)]\\&\times\int_y^\infty dz\,G_l^\pm(E;y,z)[W^\pm(z)\mp i(E-k(E))P(z)]\,\eta_l^\pm(E,z).\end{aligned} \tag{2.39}$$

Using (2.36) and the inequality

$$|E-k(E)|\leqslant\frac{m^2}{|E|},\quad |E|\geqslant m,$$

we obtain from (2.39),

$$\eta_l^\pm(E,x)=1+\int_x^\infty dy\,G_l^\pm(E;x,y)W^\pm(y)+O\left(\frac{1}{|E|^2}\right), \tag{2.40}$$

as $E\to\infty$ in $\overline{\mathbb{C}^+}$. Substituting (2.31) into (2.40) and integrating by parts we obtain (2.36). The proof of (2.37) is analogous. □

III. SCATTERING COEFFICIENTS

In this section we introduce various scattering coefficients as well as the scattering matrix for (1.2) and (1.6) and study their symmetry, asymptotic and unitarity and contractivity properties.

A. Wronskian relations and symmetry properties

Let $[f;g]=fg'-f'g$ denote the Wronskian. Then from (1.3) and (1.4) as $x\to\pm\infty$ we get

$$[f_l^\pm(E,x);f_r^\pm(E,x)]=-2iEa_l^\pm(E)=-2iEa_r^\pm(E), \tag{3.1}$$

where (3.1) holds for $E\in\overline{\mathbb{C}^+}$. Consequently, $a_l^{\pm}(E)=a_r^{\pm}(E)$, which we now denote by $a^{\pm}(E)$. From (1.8) and (1.9) we now easily obtain

$$\overline{a_s(-\overline{E})}=a_s(E),\quad E\in\overline{\mathbb{C}^+};\tag{3.2}$$

$$\overline{b_{ls}(-E)}=b_{ls}(E),\quad \overline{b_{rs}(-E)}=b_{rs}(E),\quad E\in\mathbb{R},\tag{3.3}$$

where $s=1,2$. Using (1.3), (1.4), and (2.16) we easily obtain

$$\overline{a^{\pm}(-\overline{E})}=a^{\pm}(E),\quad E\in\overline{\mathbb{C}^+};\tag{3.4}$$

$$\overline{b_l^{\pm}(-E)}=b_l^{\pm}(E),\quad \overline{b_r^{\pm}(-E)}=b_r^{\pm}(E),\quad E\in\mathbb{R}.\tag{3.5}$$

Next, if one assumes that $a^{\pm}(E)\neq 0$ and defines the transmission coefficients by $T^{\pm}(E)=a^{\pm}(E)^{-1}$, the reflection coefficients from the left by $L^{\pm}(E)=b_l^{\pm}(E)/a^{\pm}(E)$, and the reflection coefficients from the right by $R^{\pm}(E)=b_r^{\pm}(E)/a^{\pm}(E)$, then, if $P,Q\in L^1(\mathbb{R})$ [$P,Q\in L_1^1(\mathbb{R})$, respectively], the function $Ea^{\pm}(E)=E/T^{\pm}(E)$ is analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\setminus\{0\}$ [$\overline{\mathbb{C}^+}$, respectively] and the functions $Eb_l^{\pm}(E)=EL^{\pm}(E)/T^{\pm}(E)$ and $Eb_r^{\pm}(E)=ER^{\pm}(E)/T^{\pm}(E)$ are continuous in $\mathbb{R}\setminus\{0\}$ [$\mathbb{R}$, respectively]. In terms of the reflection and transmission coefficients we define the scattering matrix by

$$\mathbf{S}^{\pm}(E)=\begin{bmatrix}T^{\pm}(E) & R^{\pm}(E)\\ L^{\pm}(E) & T^{\pm}(E)\end{bmatrix}.\tag{3.6}$$

Let $E\in\mathbb{R}\setminus[-m,m]$. Then $k(-E)=-k(E)$ is real. Thus $f_l^{\pm}(E,x)$, $f_r^{\pm}(E,x)$, $f_l^{\mp}(-E,x)$ and $f_r^{\mp}(-E,x)$ all satisfy (1.2) and hence their Wronskians are independent of x. Using (1.3) and (1.4) we get

$$[f_l^{\pm}(E,x);f_l^{\mp}(-E,x)]=-2iE=-2iE[a_l^{\pm}(E)a_l^{\mp}(-E)-b_l^{\pm}(E)b_l^{\mp}(-E)],$$

$$[f_l^{\pm}(E,x);f_r^{\mp}(-E,x)]=-2iEb_r^{\mp}(-E)=2iEb_l^{\pm}(E),$$

$$[f_r^{\pm}(E,x);f_l^{\mp}(-E,x)]=-2iEb_r^{\pm}(E)=2iEb_l^{\mp}(-E),$$

$$[f_r^{\pm}(E,x);f_r^{\mp}(-E,x)]=2iE[a_r^{\pm}(E)a_r^{\mp}(-E)-b_r^{\pm}(E)b_r^{\mp}(-E)]=2iE,$$

where the behavior as $x\to+\infty$ is given first and then the behavior as $x\to-\infty$. As a result, we get

$$\mathbf{S}^{\pm}(E)^{-1}=\mathbf{S}^{\mp}(-E).\tag{3.7}$$

From (1.3), (1.4), and (3.6) we obtain

$$\begin{bmatrix}f_l^{\mp}(-E,x)\\ -f_r^{\mp}(-E,x)\end{bmatrix}=\mathbf{S}^{\pm}(E)\begin{bmatrix}f_r^{\pm}(E,x)\\ -f_l^{\pm}(E,x)\end{bmatrix}.\tag{3.8}$$

Let $E\in(-m,m)$. Then $k(-E)=k(E)$ is positive imaginary. Thus $f_l^{\pm}(E,x)$, $f_r^{\pm}(E,x)$, $f_l^{\pm}(-E,x)$ and $f_r^{\pm}(-E,x)$ all satisfy (1.2) and hence their Wronskians are independent of x. Using (1.3) and (1.4) we get

$$[f_l^{\pm}(E,x);f_l^{\pm}(-E,x)]=-2iE=-2iE[a_l^{\pm}(E)a_l^{\pm}(-E)-b_l^{\pm}(E)b_l^{\pm}(-E)],$$

$$[f_l^{\pm}(E,x);f_r^{\pm}(-E,x)]=-2iEb_r^{\pm}(-E)=2iEb_l^{\pm}(E),$$

$$[f_r^{\pm}(E,x);f_l^{\pm}(-E,x)]=-2iEb_r^{\pm}(E)=2iEb_l^{\pm}(-E),$$

$$[f_r^{\pm}(E,x);f_r^{\pm}(-E,x)]=2iE[a_r^{\pm}(E)a_r^{\pm}(-E)-b_r^{\pm}(E)b_r^{\pm}(-E)]=2iE,$$

where the behavior as $x\to+\infty$ is given first and then the behavior as $x\to-\infty$. As a result, we get

$$\mathbf{S}^{\pm}(E)^{-1}=\mathbf{S}^{\pm}(-E). \tag{3.9}$$

From (1.3), (1.4), and (3.6) we obtain

$$\begin{bmatrix} f_l^{\pm}(-E,x) \\ -f_r^{\pm}(-E,x)\end{bmatrix}=\mathbf{S}^{\pm}(E)\begin{bmatrix} f_r^{\pm}(E,x) \\ -f_l^{\pm}(E,x)\end{bmatrix}. \tag{3.10}$$

When $E\in\{-m,m\}$, the $\pm$ equations (1.2) are identical and the boundary conditions (1.3) do not distinguish between the $\pm$ versions of (1.2). It then follows that

$$f_{l1}(m,x)=f_l^+(m,x)=f_l^-(m,x),\quad f_{l1}(-m,x)=f_l^+(-m,x)=f_l^-(-m,x),$$

$$f_{r1}(m,x)=f_r^+(m,x)=f_r^-(m,x),\quad f_{r1}(-m,x)=f_r^+(-m,x)=f_r^-(-m,x),$$

which implies

$$a_1(m)=a^+(m)=a^-(m),\quad a_1(-m)=a^+(-m)=a^-(-m),$$

$$b_{l1}(m)=b_l^+(m)=b_l^-(m),\quad b_{l1}(-m)=b_l^+(-m)=b_l^-(-m),$$

$$b_{r1}(m)=b_r^+(m)=b_r^-(m),\quad b_{r1}(-m)=b_r^+(-m)=b_r^-(-m).$$

Hence, $\mathbf{S}^+(m)=\mathbf{S}^-(m)$ and $\mathbf{S}^+(-m)=\mathbf{S}^-(-m)$ are both unitary matrices, provided $a_1(m)=a_1(-m)\neq 0$. The behavior of $a_2(E)$, $b_{l2}(E)$, and $b_{r2}(E)$ as $E\to\pm m$ will be given by (3.20).

Finally, for $E\in\mathbb{R}\setminus\{-m,m\}$ and under the assumption that $a^{\pm}(E)\neq 0$ for every $E\in\mathbb{R}$, we introduce the modified scattering matrix,

$$\tilde{\mathbf{S}}(E)=\tilde{\mathbf{M}}(E)[\mathbf{S}^+(E)\oplus\mathbf{S}^-(E)]\tilde{\mathbf{M}}(E)^{-1}, \tag{3.11}$$

where

$$\tilde{\mathbf{M}}(E)=\frac{1}{2ik(E)}\begin{bmatrix} ik(E) & 0 & ik(E) & 0 \\ 0 & ik(E) & 0 & ik(E) \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix};$$

$$\tilde{\mathbf{M}}(E)^{-1}=\begin{bmatrix} 1 & 0 & ik(E) & 0 \\ 0 & 1 & 0 & ik(E) \\ 1 & 0 & -ik(E) & 0 \\ 0 & 1 & 0 & -ik(E) \end{bmatrix}.$$

Using that

$$\tilde{\mathbf{S}}(E)=\begin{bmatrix} T_1(E) & R_1(E) & -k(E)^2T_2(E) & -k(E)^2R_2(E) \\ L_1(E) & T_1(E) & -k(E)^2L_2(E) & -k(E)^2T_2(E) \\ T_2(E) & R_2(E) & T_1(E) & R_1(E) \\ L_2(E) & T_2(E) & L_1(E) & T_1(E) \end{bmatrix}, \tag{3.12}$$

we obtain from (3.8) and (3.10) the following Riemann–Hilbert problem valid for both $E\in\mathbb{R}\setminus[-m,m]$ and $E\in(-m,m)$:

$$\begin{bmatrix} f_{l1}(-E,x) \\ -f_{r1}(-E,x) \\ f_{l2}(-E,x) \\ -f_{r2}(-E,x) \end{bmatrix} = \tilde{\mathbf{S}}(E) \begin{bmatrix} f_{r1}(E,x) \\ -f_{l1}(E,x) \\ f_{r2}(E,x) \\ -f_{l2}(E,x) \end{bmatrix}. \tag{3.13}$$

One easily proves that $\tilde{\mathbf{S}}(E)^{-1} = \tilde{\mathbf{S}}(-E)$, both for $E \in (-m,m)$ and for $E \in \mathbb{R} \setminus [-m,m]$.

B. Various asymptotic properties

Theorem 3.1: *Let $P,Q \in L^1(\mathbb{R})$. Then*

$$a^{\pm}(E)e^{\pm p} = 1 + O\left(\frac{1}{|E|}\right), \quad E \to \infty \text{ in } \overline{\mathbb{C}^+}; \tag{3.14}$$

$$b_r^{\pm}(E) = O\left(\frac{1}{|E|}\right), \quad b_l^{\pm}(E) = O\left(\frac{1}{|E|}\right), \quad E \to \pm\infty, \tag{3.15}$$

where p is defined by (2.19).

Proof: From (3.1) we obtain

$$2iEa^{\pm}(E)e^{\pm p} = [2iE \pm P(x)]\eta_l^{\pm}(E,x)\eta_r^{\pm}(E,x) + \eta_l^{\pm\prime}(E,x)\eta_r^{\pm}(E,x) - \eta_l^{\pm}(E,x)\eta_r^{\pm\prime}(E,x). \tag{3.16}$$

Now (3.14) follows from (2.21), (2.22), and (3.16). Similarly, (3.15) follows with the help of

$$-2iEb_r^{\pm}(E) = e^{-2iEx \mp p \pm 2\zeta(x)}[\eta_l^{\pm}(E,x); \eta_l^{\mp}(-E,x)], \tag{3.17}$$

and the analogous expression involving $b_l^{\pm}(E)$. □

Let us now consider the low energy asymptotics of the scattering coefficients. From Ref. 17 we get the following result, depending on whether we are in the generic or in the exceptional case. We let $f_l^{\pm}(0,x)$ and $f_r^{\pm}(0,x)$ stand for the zero energy Jost functions of the usual 1-D Schrödinger equation with potential $Q(x) \mp mP(x)$ and $\gamma^{\pm}$ for the quantity given by (2.14).

Using Theorem 2.2 of Ref. 17, with $F(k) = k^2 + m^2$, $k_0 = im$, $\mathcal{S} = \{k \in \overline{\mathbb{C}^+} : |k - im| \leq m\}$ and $\mathcal{P}(k_0) = i[0,m]$, we easily obtain the following result.

Proposition 3.2: Suppose $P,Q \in L_1^1(\mathbb{R})$.

(i) In the generic case we have

$$\begin{aligned} T^{\pm}(E) &= -\frac{2iE}{[f_l^{\pm}(0,\cdot); f_r^{\pm}(0,\cdot)]} + o(E), \quad E \to 0 \text{ in } \overline{\mathbb{C}^+}, \\ L^{\pm}(E) &= -1 + o(1), \quad R^{\pm}(E) = -1 + o(1), \quad E \to 0 \text{ in } \mathbb{R}. \end{aligned} \tag{3.18}$$

(ii) In the exceptional case we have

$$T^{\pm}(0) = \frac{2\gamma^{\pm}}{\gamma^{\pm 2} + 1}, \quad L^{\pm}(0) = \frac{\gamma^{\pm 2} - 1}{\gamma^{\pm 2} + 1}, \quad R^{\pm}(0) = \frac{1 - \gamma^{\pm 2}}{\gamma^{\pm 2} + 1}. \tag{3.19}$$

Finally, we consider the behavior of the scattering coefficients as $E \to \pm m$ in $\overline{\mathbb{C}^+}$.

Propositon 3.3: Let $P \in L_1^1(\mathbb{R})$ and $Q \in L_2^1(\mathbb{R})$. Then the expressions

$$\frac{a^+(E) - a^-(E)}{k(E)}, \quad \frac{b_l^+(E) - b_l^-(E)}{k(E)}, \quad \frac{b_r^+(E) - b_r^-(E)}{k(E)}, \tag{3.20}$$

have finite limits as $E\to\pm m$, $E\in\overline{\mathbb{C}^+}$.

Proof: From (1.4), (2.3), (2.6), and (2.7) we have

$$\begin{bmatrix} a_1(E) \\ a_2(E) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2iE}\int_{-\infty}^{\infty} dy\, \Xi(E,y) \begin{bmatrix} m_{l1}(E,y) \\ m_{l2}(E,y) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2iE}\int_{-\infty}^{\infty} dy\, \Xi(E,y) \begin{bmatrix} m_{r1}(E,y) \\ m_{r2}(E,y) \end{bmatrix};$$

$$\begin{bmatrix} b_{l1}(E) \\ b_{l2}(E) \end{bmatrix} = \frac{1}{2iE}\int_{-\infty}^{\infty} dy\, e^{2iEy}\Xi(E,y) \begin{bmatrix} m_{l1}(E,y) \\ m_{l2}(E,y) \end{bmatrix};$$

$$\begin{bmatrix} b_{r1}(E) \\ b_{r2}(E) \end{bmatrix} = \frac{1}{2iE}\int_{-\infty}^{\infty} dy\, e^{-2iEy}\Xi(E,y) \begin{bmatrix} m_{r1}(E,y) \\ m_{r2}(E,y) \end{bmatrix},$$

where the limits as $E\to\pm m$ from $\overline{\mathbb{C}^+}$ exist. □

When $1/T^+(E)$ and $1/T^-(E)$ have a (necessarily common) nonzero limit as $E\to\pm m$, the next corollary is a restatement of Proposition 3.3.

Corollary 3.4: Let $P\in L_1^1(\mathbb{R})$ *and* $Q\in L_2^1(\mathbb{R})$ *and suppose* $1/T^+(E)$ *and* $1/T^-(E)$ *have a nonzero limit as* $E\to\pm m$ *in* $\overline{\mathbb{C}^+}$. *Then the expressions*

$$\frac{T^+(E)-T^-(E)}{k(E)}, \quad \frac{L^+(E)-L^-(E)}{k(E)}, \quad \frac{R^+(E)-R^-(E)}{k(E)},$$

have finite limits. Hence, $T_2(E)=[T^+(E)-T^-(E)]/2ik(E)$ *and the analogous quantities* $R_2(E)$ *and* $L_2(E)$ *are continuous in* $E\in\mathbb{R}$ *if* $T^\pm(E)$ *is continuous in* $E\in\mathbb{R}$.

Proposition 3.5: Assume $P\in L_1^1(\mathbb{R})$, $Q\in L_2^1(\mathbb{R})$, *and* $W^\pm\in L_1^1(\mathbb{R})$, *and let* $T^+(E)$ *and* $T^-(E)$ *be continuous in* $E\in\mathbb{R}$. *Then the functions* $L_1(E)$, $L_2(E)$, $k(E)^2L_2(E)$, $R_1(E)$, $R_2(E)$, *and* $k(E)^2R_2(E)$ *belong to* $L^2(\mathbb{R})$.

Proof: In view of Corollary 3.4 and the continuity of $T^\pm(E)$ in $E\in\mathbb{R}$, it suffices to study the asymptotic behavior of the above functions as $E\to\pm\infty$. From (2.36) and (2.37) we have as $E\to\pm\infty$,

$$\eta_l^{\pm\prime}(E,x) = \frac{W^\pm(x)}{2iE} + O\left(\frac{1}{|E|^2}\right), \quad \eta_r^{\pm\prime}(E,x) = -\frac{W^\mp(x)}{2iE} + O\left(\frac{1}{|E|^2}\right).$$

Using (3.17) we find

$$b_r^\pm(E) = \frac{R^\pm(E)}{T^\pm(E)} = O\left(\frac{1}{|E|^2}\right),$$

and similarly for $L^\pm(E)/T^\pm(E)$. On the other hand, using (3.16) we get

$$T^\pm(E) = e^{\pm p}\left\{1 + \frac{\int_{-\infty}^{\infty} dz\, W^\pm(z)}{2iE} + O\left(\frac{1}{|E|^2}\right)\right\},$$

whence

$$R^\pm(E) = O\left(\frac{1}{|E|^2}\right).$$

A similar asymptotic expression can be derived for $L^\pm(E)$. This expression implies that $L_1(E)$, $L_2(E)$, and $k(E)^2L_2(E)$ belong to $L^2(\mathbb{R})$. □

C. Unitarity and contractivity properties

Let $E\in(-m,m)$. Then (1.2) is a pair of 1-D Schrödinger equations with real potentials and hence the scattering matrix $\mathbf{S}^{\pm}(E)$ is unitary (cf. Refs. 15 and 16). As a result, the reflection and transmission coefficients $R^{\pm}(E)$, $L^{\pm}(E)$, and $T^{\pm}(E)$ are continuous in $E\in(-m,m)$.

Let $E\in\mathbb{R}\setminus[-m,m]$. Observe that if $\psi(E,x)$ is a solution of the $\pm$ version of (1.2) and $\varphi(E,x)$ of the $\mp$ version of (1.2), then

$$\frac{d}{dx}[\psi(E,x);\varphi(E,x)]=\mp 2ik(E)P(x)\psi(E,x)\varphi(E,x). \tag{3.21}$$

Hence two expressions of the Wronskian of $\psi(E,x)$ and $\varphi(E,x)$ can be found by examining their value as $x\to\pm\infty$ and integrating with respect to x. Using (2.16), (3.4), (3.5), and (3.21) we get

$$[f_l^{\pm}(E,x);f_l^{\pm}(-E,x)]=\begin{cases}-2iE\pm 2ik(E)\int_x^{\infty}dy\,P(y)|f_l^{\pm}(E,y)|^2;\\ -2iE[|a^{\pm}(E)|^2-|b_l^{\pm}(E)|^2]\mp 2ik(E)\int_{-\infty}^{x}dy\,P(y)|f_l^{\pm}(E,y)|^2,\end{cases} \tag{3.22}$$

$$[f_l^{\pm}(E,x);f_r^{\pm}(-E,x)]=\begin{cases}2iEb_l^{\pm}(E)\mp 2ik(E)\int_{-\infty}^{x}dy\,P(y)f_l^{\pm}(E,y)\overline{f_r^{\pm}(E,y)};\\ -2iE\overline{b_r^{\pm}(E)}\pm 2ik(E)\int_x^{\infty}dy\,P(y)f_l^{\pm}(E,y)\overline{f_r^{\pm}(E,y)},\end{cases} \tag{3.23}$$

$$[f_r^{\pm}(E,x);f_r^{\pm}(-E,x)]=\begin{cases}2iE\mp 2ik(E)\int_{-\infty}^{x}dy\,P(y)|f_r^{\pm}(E,y)|^2;\\ 2iE[|a^{\pm}(E)|^2-|b_r^{\pm}(E)|^2]\pm 2ik(E)\int_x^{\infty}dy\,P(y)|f_r^{\pm}(E,y)|^2.\end{cases} \tag{3.24}$$

Subtracting the two right-hand sides of each of (3.22)–(3.24), we get

$$-1+|a^{\pm}(E)|^2-|b_l^{\pm}(E)|^2=\mp\frac{k(E)}{E}\int_{-\infty}^{\infty}dy\,P(y)|f_l^{\pm}(E,y)|^2; \tag{3.25}$$

$$-b_l^{\pm}(E)-\overline{b_r^{\pm}(E)}=\mp\frac{k(E)}{E}\int_{-\infty}^{\infty}dy\,P(y)f_l^{\pm}(E,y)\overline{f_r^{\pm}(E,y)}, \tag{3.26}$$

$$-1+|a^{\pm}(E)|^2-|b_r^{\pm}(E)|^2=\mp\frac{k(E)}{E}\int_{-\infty}^{\infty}dy\,P(y)|f_r^{\pm}(E,y)|^2. \tag{3.27}$$

From (3.25) and (3.27) it is clear that $a^{\pm}(E)\neq 0$ when $(\mp P(x))\geqslant 0$. In that case we define the matrix

$$\mathbf{W}^{\pm}(E)=\mp\begin{bmatrix}\int_{-\infty}^{\infty}dyP(y)|f_l^{\pm}(E,y)|^2 & \int_{-\infty}^{\infty}dyP(y)f_r^{\pm}(E,y)\overline{f_l^{\pm}(E,y)}\\ \int_{-\infty}^{\infty}dyP(y)\overline{f_r^{\pm}(E,y)}f_l^{\pm}(E,y) & \int_{-\infty}^{\infty}dyP(y)|f_r^{\pm}(E,y)|^2\end{bmatrix},$$

and, provided $a^{\pm}(E)\neq 0$, derive the identity

$$\frac{k(E)}{E}|T^{\pm}(E)|^2\mathbf{W}^{\pm}(E)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\mathbf{S}^{\pm}(E)^{\dagger}\mathbf{S}^{\pm}(E),$$

which is nonnegative selfadjoint if $(\mp P(x))\geqslant 0$. Here † denotes the conjugate transpose. Similarly, one proves that if $(\mp P(x))\leqslant 0$ and the transmission coefficient $T^{\pm}(E)$ is well-defined, the scattering matrix $\mathbf{S}^{\pm}(E)$ has a contractive inverse.

D. Discrete eigenvalues

The discrete eigenvalues of the pair of modified Schrödinger equations (1.2) coincide with those of the system (1.6). They form a finite or countably infinite subset of $\mathbb{C}^{+}$ of eigenvalues of finite algebraic multiplicity. The geometric multiplicity of the eigenvalues of either of the equations (1.2) is one, while that of (1.6) is at most two. They can only accumulate in a bounded interval of the real line, but not at points of $(-m,m)$. Accumulation as $E\rightarrow\infty$ is impossible because of (3.14). Accumulation at points of $(-m,m)$ is impossible, because the scattering matrix $\mathbf{S}^{\pm}(E)$ is unitary if $E\in(-m,m)$.

The discrete eigenvalues are symmetrically located with respect to the imaginary axis, where the geometric and algebraic multiplicities of an eigenvalue at E_0 coincide with those at $-\overline{E_0}$. This follows directly from (3.4)–(3.5). The net result is that the residues of $iT^{\pm}(E)$ at E_0 and $-\overline{E_0}$ are complex conjugates.

For the problem (1.2) with $m=0$, the properties of the discrete spectrum have been discussed in detail in Ref. 6. Many of these results also follow from spectral properties of certain operator pencils (cf. Ref. 19). If $m>0$, most of those results are expected to go through, albeit in a slightly different form.

IV. MARCHENKO EQUATIONS

In this section we derive the Marchenko integral equations leading to the solution of the inverse scattering problem.

A. Fourier transformation properties

Let us apply the method of Ref. 13 to derive Marchenko integral equations to solve the inverse scattering problem. We begin by deriving some integral representations for the (transformed) Jost solutions.

Theorem 4.1: *Assume* $P\in L_1^1(\mathbb{R})$, $Q\in L_2^1(\mathbb{R})$, *and* $W^{\pm}\in L_1^1(\mathbb{R})$. *Then the Jost solutions* $f_{rs}(E,x)$ *and* $f_{ls}(E,s)$ $(s=1,2)$ *can be represented as follows:*

$$f_{r1}(E,x)=e^{-iEx}\cosh(p-\zeta(x))+\int_{-\infty}^{x}dt\,K_{r1}(x,t)e^{-iEt}, \tag{4.1}$$

$$f_{r2}(E,x)=\int_{-\infty}^{x}dt\,K_{r2}(x,t)e^{-iEt}, \tag{4.2}$$

$$f_{l1}(E,x)=e^{iEx}\cosh(\zeta(x))+\int_{x}^{\infty}dt\,K_{l1}(x,t)e^{iEt}, \tag{4.3}$$

$$f_{l2}(E,x)=\int_{x}^{\infty}dt\,K_{l2}(x,t)e^{iEt}, \tag{4.4}$$

where $K_{rs}^{\pm}(x,t)$ *and* $K_{ls}^{\pm}(x,t)$ $(s=1,2)$ *are independent of* E *and belong to* $L^2(\mathbb{R})$ *as functions of* t *when* $x\in\mathbb{R}$ *is fixed.*

Proof: Using (2.17), (2.18), (2.36), and (2.37) it follows that $\tilde{f}_{l1}(E,x)=f_{l1}(E,x)-e^{iEx}\cosh(\zeta(x))$ and $\tilde{f}_{r1}(E,x)=f_{r1}(E,x)-e^{-iEx}\cosh(p-\zeta(x))$, as well as $\tilde{f}_{l2}(E,x)=f_{l2}(E,x)$ and $\tilde{f}_{r2}(E,x)=f_{r2}(E,x)$ belong to $L^2(\mathbb{R})$ as functions of E for fixed $x\in\mathbb{R}$. Further, these functions multiplied by $(\log(|E|+2))^{1/2}$ belong to $L^2(\mathbb{R})$. So Plancherel's theorem (cf. Ref. 20, Theorems 48 and 63) implies the existence of the integrals,

$$K_{rs}(x,t)\underset{a\to+\infty}{\overset{l.i.m.}{=}}\frac{1}{2\pi}\int_{-a}^{a}dE\,\tilde{f}_{rs}(E,x)e^{iEt},\quad s=1,2,$$

$$K_{ls}(x,t)\underset{a\to+\infty}{\overset{l.i.m.}{=}}\frac{1}{2\pi}\int_{-a}^{a}dE\,\tilde{f}_{ls}(E,x)e^{-iEt},\quad s=1,2.$$

It is clear that $K_{rs}(x,t)$ and $K_{ls}(x,t)$ $(s=1,2)$ belong to $L^2(\mathbb{R})$ as functions of t for every $x\in\mathbb{R}$ (cf. Ref. 20, Theorems 48 and 63).

Due to (1.7) and Theorem 2.1, the functions $f_{rs}(E,x)$ and $f_{ls}(E,x)$ $(s=1,2)$ are analytic in $E\in\mathbb{C}^+$. Moreover, there exists $C>0$ (depending on $x\in\mathbb{R}$) such that for $s=1,2$,

$$|f_{rs}(E,x)|\leqslant Ce^{x\,\mathrm{Im}\,E},\quad |f_{ls}(E,x)|\leqslant Ce^{-x\,\mathrm{Im}\,E},$$

for all $x\in\mathbb{R}$. From (2.37) we obtain

$$\int_{-\infty}^{\infty}dt\,|\tilde{f}_{r1}(t+i\,\mathrm{Im}\,E,x)|^2=O(e^{-2x\,\mathrm{Im}\,E}).$$

Similar estimates hold for $K_{r2}(E,x)$ and for $K_{ls}(E,x)$ $(s=1,2)$. Hence we may apply Titchmarsh's theorem (cf. Ref. 20, Theorem 96) and obtain

$$K_{r1}(x,t)=K_{r2}(x,t)=0,\quad t>x,$$

$$K_{l1}(x,t)=K_{l2}(x,t)=0,\quad t<x.$$

This proves the representations (4.1)–(4.4). □

Using (2.18), (2.19), and (2.37), we obtain

$$\begin{aligned}\tilde{f}_{r1}(E,x)&=\frac{ie^{-iEx}}{4E}\left(e^{-p+\zeta(x)}\int_{-\infty}^{x}dz\,W^+(z)+e^{p-\zeta(x)}\int_{-\infty}^{x}dz\,W^-(z)\right)+O\left(\frac{1}{|E|^2}\right)\\&=\frac{ie^{-iEx}}{4(E+i\chi)}\left(e^{-p+\zeta(x)}\int_{-\infty}^{x}dz\,W^+(z)+e^{p-\zeta(x)}\int_{-\infty}^{x}dz\,W^-(z)\right)+O\left(\frac{1}{|E|^2}\right),\end{aligned}$$

where χ is an arbitrary positive number. Its Fourier transform is of the form

$$K_{r1}(x,t)=\left[\frac{e^{-p+\zeta(x)}}{4}\int_{-\infty}^{x}dz\,W^+(z)+\frac{e^{p-\zeta(x)}}{4}\int_{-\infty}^{x}dz\,W^-(z)\right]e^{-\chi(x-t)}\theta(x-t)+M_{r1}(x,t),$$

where $\theta(z)$ is the Heaviside function given by

$$\theta(z)=\begin{cases}0, & \text{for } z<0,\\ 1, & \text{for } z>0,\end{cases}$$

$M_{r1}(x,t)$ is continuous in $t\in\mathbb{R}$ for fixed $x\in\mathbb{R}$ and there exists the partial derivative $(\partial M_{r1}(x,t)/\partial t)\in L^2(\mathbb{R})$ (cf. Ref. 20, the beginning of Sec. 6.13, before Theorem 128). Hence, $K_{r1}(x,t)$ has a jump discontinuity at $x=t$. Taking into account the identity $K_{r1}(x,x+0)=0$, we obtain

$$K_{r1}(x,x-0)=\frac{1}{4}\left(e^{-p+\zeta(x)}\int_{-\infty}^{x}dz\,W^{+}(z)+e^{p-\zeta(x)}\int_{-\infty}^{x}dz\,W^{-}(z)\right). \tag{4.5}$$

In the same way we obtain

$$K_{r2}(x,x-0)=\sinh(p-\zeta(x)). \tag{4.6}$$

Using (4.5) and (4.6) we now compute $Q(x)$ and $P(x)$ from $K_{r1}(x,x-0)$ and $K_{r2}(x,x-0)$. In fact, we obtain

$$P(x)=2\frac{d}{dx}\log(K_{r2}(x,x-0)+(K_{r2}(x,x-0)^2+1)^{1/2}), \tag{4.7}$$

$$Q(x)=2\frac{d}{dx}\left(\frac{K_{r1}(x,x-0)}{\cosh\frac{1}{2}\int_{-\infty}^{x}dz\,P(z)}-\frac{P(x)}{4}\tanh\left(\frac{1}{2}\int_{-\infty}^{x}dz\,P(z)\right)\right)+\frac{P(x)^2}{4}. \tag{4.8}$$

In the same way we derive

$$K_{l2}(x,x+0)=\sinh(\zeta(x)),$$

$$P(x)=-2\frac{d}{dx}\log(K_{l2}(x,x-0)+(K_{l2}(x,x-0)^2+1)^{1/2}), \tag{4.9}$$

$$Q(x)=-2\frac{d}{dx}\left(\frac{K_{l1}(x,x+0)}{\cosh\frac{1}{2}\int_{x}^{\infty}dz\,P(z)}-\frac{P(x)}{4}\tanh\left(\frac{1}{2}\int_{x}^{\infty}dz\,P(z)\right)\right)+\frac{P(x)^2}{4}. \tag{4.10}$$

B. Marchenko equations without bound states

Let us assume that $T^{+}(E)$ and $T^{-}(E)$ are both continuous in $E\in\mathbb{R}$. Before deriving the two pairs of Marchenko integral equations, we introduce the two sets of integral kernels as follows:

$$F_{l1}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\,L_1(E)e^{iEx}=\int_{-\infty}^{\infty}\frac{dE}{4\pi}[L^{+}(E)+L^{-}(E)]e^{iEx}, \tag{4.11}$$

$$F_{l2}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\,L_2(E)e^{iEx}=\int_{-\infty}^{\infty}\frac{dE}{4\pi}\frac{L^{+}(E)-L^{-}(E)}{i\,k(E)}e^{iEx}, \tag{4.12}$$

$$\begin{aligned}F_{l3}(x)&=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\,k(E)^2L_2(E)e^{iEx}\\&=i\int_{-\infty}^{\infty}\frac{dE}{4\pi}k(E)[L^{+}(E)-L^{-}(E)]e^{iEx},\end{aligned} \tag{4.13}$$

as well as

$$F_{r1}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\,R_1(E)e^{iEx}=\int_{-\infty}^{\infty}\frac{dE}{4\pi}[R^{+}(E)+R^{-}(E)]e^{iEx}, \tag{4.14}$$

$$F_{r2}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\,R_2(E)e^{iEx}=\int_{-\infty}^{\infty}\frac{dE}{4\pi}\,\frac{R^+(E)-R^-(E)}{i\,k(E)}e^{iEx}, \tag{4.15}$$

$$\begin{aligned}F_{r3}(x)&=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dE\,k(E)^2R_2(E)e^{iEx}\\&=i\int_{-\infty}^{\infty}\frac{dE}{4\pi}k(E)[R^+(E)-R^-(E)]e^{iEx}.\end{aligned} \tag{4.16}$$

Then the integrals (4.11)–(4.13) and (4.14)–(4.16) exist as a result of Proposition 3.5 and the continuity of $T^+(E)$ and $T^-(E)$ in $E\in\mathbb{R}$.

Next, we introduce the unknown functions $B_{rs}(x,y)$ and $B_{ls}(x,y)$ ($x\in\mathbb{R}$, $y>0$, $s=1,2$) by

$$B_{rs}(x,y)=\frac{K_{rs}(x,x-y)}{\cosh(p-\zeta(x))},\quad B_{ls}(x,y)=\frac{K_{ls}(x,x+y)}{\cosh(\zeta(x))}, \tag{4.17}$$

and write (4.1)–(4.4) in the form

$$f_{rs}(E,x)=e^{-iEx}\cosh(p-\zeta(x))\left[\delta_{s,1}+\int_0^{\infty}dy\,e^{iEy}B_{rs}(x,y)\right],$$

$$f_{ls}(E,x)=e^{iEx}\cosh(\zeta(x))\left[\delta_{s,1}+\int_0^{\infty}dy\,e^{iEy}B_{ls}(x,y)\right],$$

where $s=1,2$.

Starting from the two pairs of equations [cf. (3.13)],

$$\begin{aligned}&f_{r1}(-E,x)+L_1(E)f_{r1}(E,x)-k(E)^2L_2(E)f_{r2}(E,x)\\&\quad=T_1(E)f_{l1}(E,x)-k(E)^2T_2(E)f_{l2}(E,x),\end{aligned} \tag{4.18}$$

$$\begin{aligned}&f_{r2}(-E,x)+L_2(E)f_{r1}(E,x)+L_1(E)f_{r2}(E,x)\\&\quad=T_2(E)f_{l1}(E,x)+T_1(E)f_{l2}(E,x),\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&f_{l1}(-E,x)+R_1(E)f_{l1}(E,x)-k(E)^2R_2(E)f_{l2}(E,x)\\&\quad=T_1(E)f_{r1}(E,x)-k(E)^2T_2(E)f_{r2}(E,x),\end{aligned} \tag{4.20}$$

$$\begin{aligned}&f_{l2}(-E,x)+R_2(E)f_{l1}(E,x)+R_1(E)f_{l2}(E,x)\\&\quad=T_2(E)f_{r1}(E,x)+T_1(E)f_{r2}(E,x),\end{aligned} \tag{4.21}$$

and Fourier transforming the contributions to these equations that are analytic in $\mathbb{C}^-$ and vanish at infinity while taking into account (4.11)–(4.17), we obtain the two pairs of coupled Marchenko equations,

$$\begin{aligned}&B_{r1}(x,y)+\int_0^{\infty}dz\,[F_{l1}(y+z-2x)B_{r1}(x,z)+F_{l3}(y+z-2x)B_{r2}(x,z)]\\&\quad=-F_{l1}(y-2x),\end{aligned} \tag{4.22}$$

$$B_{r2}(x,y)+\int_0^\infty dz\,[F_{l2}(y+z-2x)B_{r1}(x,z)+F_{l1}(y+z-2x)B_{r2}(x,z)]$$
$$=-F_{l2}(y-2x), \tag{4.23}$$

and

$$B_{l1}(x,y)+\int_0^\infty dz\,[F_{r1}(y+z+2x)B_{l1}(x,z)+F_{r3}(y+z+2x)B_{l2}(x,z)]$$
$$=-F_{l1}(y+2x), \tag{4.24}$$

$$B_{l2}(x,y)+\int_0^\infty dz\,[F_{r2}(y+z+2x)B_{l1}(x,z)+F_{r1}(y+z+2x)B_{l2}(x,z)]$$
$$=-F_{r2}(y+2x). \tag{4.25}$$

In deriving (4.22)–(4.25), we have assumed the absence of the discrete spectrum of the two equations (1.2) and hence the analyticity of $T_1(E)$ and $T_2(E)$ in $\mathbb{C}^+$.

C. Marchenko equations with bound states

When one of the two equations (1.2) has a discrete spectrum, the derivation of the Marchenko equations (4.22)–(4.25) should be modified, since the right-hand sides of (4.18)–(4.21) may no longer vanish. To simplify the discussion, we make the following assumptions.

(1) $T^+(E)$ and $T^-(E)$ are continuous in $E\in\mathbb{R}$.
(2) The number of poles of $T^+(E)$ and $T^-(E)$ in $\mathbb{C}^+$ is finite [denote the poles of either of $T^\pm(E)$ in $\mathbb{C}^+$ by E_j, where $j=1,\ldots,\mathcal{N}$].
(3) The poles of $T^+(E)$ and $T^-(E)$ in $\mathbb{C}^+$ are simple; we write $it_j^\pm$ for the residue of $T^\pm(E)$ at $E=E_j$ $(j=1,\ldots,\mathcal{N})$. We put $t_{j1}=[t_j^+ + t_j^-]/2$ and $t_{j2}=[t_j^+ - t_j^-]/2ik(E_j)$.
(4) We remark that $t_j^\pm$, t_{j1}, t_{j2}, and $ik(E_j)$ are real if E_j is imaginary. Quantities $t_j^\pm$, t_{j1}, t_{j2}, and $ik(E_j)$ corresponding to eigenvalues symmetrically located with respect to the imaginary axis are complex conjugates. These properties are immediate from the observations made in Sec. III D.
(5) Using the terminology of Ref. 16 and recalling that the eigenvalues of either of (1.2) have geometric multiplicity one, we first introduce the norming constants,

$$C_j^\pm=\frac{f_l^\pm(E_j,x)}{f_r^\pm(E_j,x)},\quad j=1,\ldots,\mathcal{N}_j^\pm .$$

Then one easily verifies that

$$f_{l1}(E_j,x)=C_{j1}f_{r1}(E_j,x)-k_j^2C_{j2}f_{r2}(E_j,x),$$

$$f_{l2}(E_j,x)=C_{j2}f_{r1}(E_j,x)+C_{j1}f_{r2}(E_j,x),$$

where $k_j=k(E_j)$ and

$$C_{j1}=\frac{C_j^+ + C_j^-}{2},\quad C_{j2}=\frac{C_j^+ - C_j^-}{2ik_j}.$$

Calculating the residues of the expressions on the right-hand sides of (4.18)–(4.21) at $E=E_j$ in $\mathbb{C}^+$, we obtain

$$[t_{j1}C_{j1}-k_j^2t_{j2}C_{j2}]f_{r1}(E_j,x)-k_j^2[t_{j1}C_{j2}+t_{j2}C_{j1}]f_{r2}(E_j,x), \tag{4.26}$$

$$[t_{j2}C_{j1}+t_{j1}C_{j2}]f_{r1}(E_j,x)+[-k_j^2t_{j2}C_{j2}+t_{j1}C_{j1}]f_{r2}(E_j,x), \tag{4.27}$$

$$[t_{j1}D_{j1}-k_j^2t_{j2}D_{j2}]f_{r1}(E_j,x)-k_j^2[t_{j1}D_{j2}+t_{j2}D_{j1}]f_{r2}(E_j,x), \tag{4.28}$$

$$[t_{j2}D_{j1}+t_{j1}D_{j2}]f_{r1}(E_j,x)+[-k_j^2t_{j2}D_{j2}+t_{j1}D_{j1}]f_{r2}(E_j,x), \tag{4.29}$$

multiplied by the imaginary unit i. Here

$$f_{r1}(E_j,x)=D_{j1}f_{l1}(E_j,x)-k_j^2D_{j2}f_{l2}(E_j,x),$$

$$f_{r2}(E_j,x)=D_{j2}f_{l1}(E_j,x)+D_{j1}f_{l2}(E_j,x),$$

where we note that

$$\begin{bmatrix} D_{j1} & -k_j^2D_{j2} \\ D_{j2} & D_{j1} \end{bmatrix}=\begin{bmatrix} C_{j1} & -k_j^2C_{j2} \\ C_{j2} & C_{j1} \end{bmatrix}^{-1}.$$

We remark that $C_j^{\pm}$, C_{j1}, C_{j2}, D_{j1}, D_{j2} and ik_j are real if E_j is imaginary. Quantities $t_j^{\pm}$, C_{j1}, C_{j2}, D_{j1}, D_{j2}, and ik_j corresponding to eigenvalues symmetrically located with respect to the imaginary axis are complex conjugates. These properties are immediate from (2.16).

We now recall that in order to compute the left-hand side minus the right-hand side of (4.22) and (4.23), we have to single out the contributions to (4.18) and (4.19) that are analytic in $\mathbb{C}^-$ and vanish at infinity and apply the operation $1/2\pi\int_{-\infty}^{\infty}dE\, e^{iE(y-x)}[\cosh(p-\zeta(x))]^{-1}$ to them. Applying the same procedure to the right-hand sides of (4.18) and (4.19) and using (4.17), (4.26), and (4.27), we obtain

$$-\sum_{j=1}^{\mathcal{N}} e^{iE_j(y-2x)}\left(A_{lj1}+\int_0^\infty dz\, e^{iE_jz}[A_{lj1}B_{r1}(x,z)-k_j^2A_{lj2}B_{r2}(x,z)]\right),$$

$$-\sum_{j=1}^{\mathcal{N}} e^{iE_j(y-2x)}\left(A_{lj2}+\int_0^\infty dz\, e^{iE_jz}[A_{lj2}B_{r1}(x,z)-k_j^2A_{lj1}B_{r2}(x,z)]\right),$$

where

$$A_{lj1}=t_{j1}C_{j1}-k_j^2t_{j2}C_{j2},\quad A_{lj2}=t_{j1}C_{j2}+t_{j2}C_{j1}.$$

Introducing the modified Marchenko kernel functions,

$$\tilde{F}_{ls}(x)=F_{ls}(x)+\sum_{j=1}^{\mathcal{N}} A_{ljs}e^{iE_jx},\quad s=1,2,3,$$

where $A_{lj3}=-k_j^2A_{lj2}$, we arrive at the coupled Marchenko integral equations,

$$B_{r1}(x,y)+\int_0^\infty dz\,[\tilde{F}_{l1}(y+z-2x)B_{r1}(x,z)+\tilde{F}_{l3}(y+z-2x)B_{r2}(x,z)]$$
$$=-\tilde{F}_{l1}(y-2x), \tag{4.30}$$

$$B_{r2}(x,y)+\int_0^\infty dz\,[\tilde{F}_{l2}(y+z-2x)B_{r1}(x,z)+\tilde{F}_{l1}(y+z-2x)B_{r2}(x,z)]$$
$$=-\tilde{F}_{l2}(y-2x). \tag{4.31}$$

In an analogous way we obtain the coupled Marchenko integral equations

$$B_{l1}(x,y)+\int_0^\infty dz\,[\tilde{F}_{r1}(y+z+2x)B_{l1}(x,z)+\tilde{F}_{r3}(y+z+2x)B_{l2}(x,z)]$$
$$=-\tilde{F}_{r1}(y+2x), \tag{4.32}$$

$$B_{l2}(x,y)+\int_0^\infty dz\,[\tilde{F}_{r2}(y+z+2x)B_{l1}(x,z)+\tilde{F}_{r1}(y+z+2x)B_{l2}(x,z)]$$
$$=-\tilde{F}_{r2}(y+2x), \tag{4.33}$$

where

$$\tilde{F}_{rs}(x)=F_{rs}(x)+\sum_{j=1}^{\mathcal{N}} A_{rjs}e^{iE_jx}, \quad s=1,2,3,$$

$$A_{rj1}=t_{j1}D_{j1}-k_j^2t_{j2}D_{j2}, \quad A_{rj2}=t_{j1}D_{j2}+t_{j2}D_{j1},$$

and $A_{rj3}=-k_j^2A_{rj2}$. Using the symmetry statements made before in this subsection, one easily proves that $\tilde{F}_{ls}(x)$ and $\tilde{F}_{rs}(x)$ $(s=1,2,3)$ are real functions.

When $T^+(E)$ and $T^-(E)$ both have a finite number of poles and some of them are multiple poles (but otherwise the first assumption is fulfilled), (4.30)–(4.33) can be derived using a generalization of the notion of norming constant given in Ref. 7, but with more complicated auxiliary kernel functions $\tilde{F}_{ls}(y)$ and $\tilde{F}_{rs}(y)$ $(s=1,2,3)$.

We now state the main result. The first part is immediate from (4.7)–(4.8) and (4.17). The second part follows from the second (4.17) and expressions involving $K_{ls}(x,t)$ and $B_{ls}(x,y)$ $(s=1,2)$ analogous to (4.7)–(4.8).

Theorem 4.2: *Suppose $P\in L_1^1(\mathbb{R})$, $Q\in L_2^1(\mathbb{R})$, and $W^\pm\in L_1^1(\mathbb{R})$, and let conditions (1)–(3) stated at the beginning of Sec. IV C be fulfilled. Then if $B_{rs}(x,\cdot)$ $(s=1,2)$ are the solutions of the Marchenko equations (4.30) and (4.31) and $B_{r2}(x,0^+)\in(-1,1)$, the potentials $Q(x)$ and $P(x)$ are given by*

$$P(x)=\frac{d}{dx}\log\frac{1+B_{r2}(x,0^+)}{1-B_{r2}(x,0^+)}; \tag{4.34}$$

$$Q(x)=2\frac{d}{dx}\left[B_{r1}(x,0^+)-\frac{P(x)}{4}\tanh\left(\frac{1}{2}\int_{-\infty}^x dz\,P(z)\right)\right]+\frac{P(x)^2}{4}. \tag{4.35}$$

Similarly, if $B_{ls}(x,\cdot)$ $(s=1,2)$ are the solutions of the Marchenko equations (4.32) and (4.33) and $B_{l2}(x,0^+)\in(-1,1)$, then the potentials $Q(x)$ and $P(x)$ are given by

$$P(x)=\frac{d}{dx}\log\frac{1-B_{l2}(x,0^+)}{1+B_{l2}(x,0^+)}; \tag{4.36}$$

$$Q(x)=-2\frac{d}{dx}\left[B_{l1}(x,0^+)-\frac{P(x)}{4}\tanh\left(\frac{1}{2}\int_x^\infty dz\,P(z)\right)\right]+\frac{P(x)^2}{4}. \tag{4.37}$$

V. SOLVABILITY OF THE MARCHENKO EQUATIONS

In this section we establish the compactness of the Marchenko integral operators and relate the unique solvability of the (pairs of) Marchenko integral equations to the canonical factorizability of a matrix function.

Theorem 5.1: *Suppose $P \in L_1^1(\mathbb{R})$, $Q \in L_2^1(\mathbb{R})$, and $W^{\pm} \in L_1^1(\mathbb{R})$, and let conditions (1)–(3) stated at the beginning of Sec. IV C be fulfilled. Then the integral operators arising from the Marchenko integral equations (4.30)–(4.33) are compact on $L^2(\mathbb{R}^+)$.*

Proof: All of these integral operators have the form

$$(Kg)(y) = \int_0^\infty dz\, F(y+z)g(z), \quad y>0,$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\, \Phi(E)e^{iEx},$$

for some function $\Phi(E)$ that is continuous in $E \in \mathbb{R}$ and vanishes as $E \to \pm\infty$. Such integral operators are Hankel operators with continuous symbol and as such compact on $L^2(\mathbb{R})$ (cf. Refs. 21 and 22). □

In order to derive sufficient conditions for the unique solvability of the Marchenko equations (4.30)–(4.31) or the Marchenko equations (4.32)–(4.33), we define the quantities

$$\hat{R}_s(E) = \int_{-\infty}^{\infty} dz\, e^{-iEz}\tilde{F}_{rs}(z), \quad \hat{L}_s(E) = \int_{-\infty}^{\infty} dz\, e^{-iEz}\tilde{F}_{ls}(z),$$

where $s=1,2,3$. If neither of (1.2) has any eigenvalues, we have

$$\hat{R}_1(E) = R_1(E), \quad \hat{R}_2(E) = R_2(E), \quad \hat{R}_3(E) = -k(E)^2 R_2(E),$$

$$\hat{L}_1(E) = L_1(E), \quad \hat{L}_2(E) = L_2(E), \quad \hat{L}_3(E) = -k(E)^2 L_2(E).$$

Introducing the functions

$$B_{rs}^{\pm}(E,x) = \pm\int_0^{\pm\infty} dy\, B_{rs}(x,y)e^{iEy}, \quad B_{ls}^{\pm}(E,x) = \pm\int_0^{\pm\infty} dy\, B_{ls}(x,y)e^{iEy},$$

where $s=1,2$, by Fourier transformation we obtain from (4.30)–(4.31) the Riemann–Hilbert problem,

$$\begin{bmatrix} I & 0 \\ \mathbf{F}_l(-E,x) & I \end{bmatrix}\begin{bmatrix} \mathbf{X}^+(E,x) \\ \mathbf{X}^-(-E,x) \end{bmatrix} + \begin{bmatrix} I & \mathbf{F}_l(E,x) \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbf{X}^-(E,x) \\ \mathbf{X}^+(-E,x) \end{bmatrix} = \begin{bmatrix} \mathbf{Y}(-E,x) \\ \mathbf{Y}(E,x) \end{bmatrix}, \tag{5.1}$$

where I denotes the 2×2 identity matrix and

$$\mathbf{X}^{\pm}(E,x) = \begin{bmatrix} B_{r1}^{\pm}(E,x) \\ B_{r2}^{\pm}(E,x) \end{bmatrix}, \quad \mathbf{Y}(E,x) = -e^{-2iEx}\begin{bmatrix} \hat{L}_1(E) \\ \hat{L}_2(E) \end{bmatrix},$$

$$\mathbf{F}_l(E,x) = e^{-2iEx}\begin{bmatrix} \hat{L}_1(E) & \hat{L}_3(E) \\ \hat{L}_2(E) & \hat{L}_1(E) \end{bmatrix}.$$

In the same way we define

$$\mathbf{F}_r(E,x)=e^{2iEx}\begin{bmatrix}\hat{R}_1(E) & \hat{R}_3(E)\\ \hat{R}_2(E) & \hat{R}_1(E)\end{bmatrix}.$$

In analogy with (5.1), we derive the Riemann–Hilbert problem,

$$\begin{bmatrix} I & 0\\ \mathbf{F}_l(-E,x) & I\end{bmatrix}\begin{bmatrix}\mathbf{X}^+(E,x)\\ -\mathbf{X}^-(-E,x)\end{bmatrix}+\begin{bmatrix} I & \mathbf{F}_l(E,x)\\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbf{X}^-(E,x)\\ -\mathbf{X}^+(-E,x)\end{bmatrix}=\begin{bmatrix}\mathbf{Y}(-E,x)\\ -\mathbf{Y}(E,x)\end{bmatrix},$$

from the system of integral equations obtained from (4.30) and (4.31) by replacing the kernels $\hat{F}_{ls}$ with $-\hat{F}_{ls}$.

Next, by a (right) *canonical factorization* of a matrix function $W(E)$ defined for $E\in\mathbb{R}$ we mean a factorization of the form

$$W(E)=W^-(E)W^+(E), \tag{5.2}$$

where both $W^{\pm}(E)$ and $W^{\pm}(E)^{-1}$ are continuous in $E\in\overline{\mathbb{C}^{\pm}}$, are analytic in $E\in\mathbb{C}^{\pm}$, and have a limit as $E\to\infty$ in $\overline{\mathbb{C}^{\pm}}$. Replacing (5.2) with $W(E)=W^+(E)W^-(E)$, we get the definition of a left canonical factorization.

The following theorem easily follows using the methods employed in Refs. 23 and 24. Such methods were applied to inverse scattering before in Ref. 7.

Theorem 5.2: *Suppose* $P\in L_1^1(\mathbb{R})$, $Q\in L_2^1(\mathbb{R})$ *and* $W^{\pm}\in L_1^1(\mathbb{R})$, *and let conditions (1)–(3) stated at the beginning of Sec. IV C be fulfilled. Then, for fixed* $x\in\mathbb{R}$, *the system of Marchenko integral equations (4.30) and (4.31) and the system of integral equations obtained from them by replacing the kernels* $\tilde{F}_{ls}$ *with* $-\tilde{F}_{ls}$ *both have a unique solution if and only if the* 4×4 *matrix function*

$$\begin{bmatrix} I-\mathbf{F}_l(E,x)\mathbf{F}_l(-E,x) & -\mathbf{F}_l(E,x)\\ \mathbf{F}_l(-E,x) & I\end{bmatrix} \tag{5.3}$$

has a (right) canonical factorization. Similarly, for fixed $x\in\mathbb{R}$, *the system of Marchenko integral equations (4.32) and (4.33) and the system of integral equations obtained from them by replacing the kernels* $\tilde{F}_{rs}$ *with* $-\tilde{F}_{rs}$ *both have a unique solution if and only if the* 4×4 *matrix function*

$$\begin{bmatrix} I-\mathbf{F}_r(E,x)\mathbf{F}_r(-E,x) & -\mathbf{F}_r(E,x)\\ \mathbf{F}_r(-E,x) & I\end{bmatrix} \tag{5.4}$$

has a (right) canonical factorization.

In Ref. 7 the Marchenko equations are simple enough to allow for a representation of the 4×4 matrix functions in (5.3) and (5.4) as the direct sum of two 2×2 matrix functions (one being the adjoint of the other) multiplied on either side by constant nonsingular matrices. As a result, in Ref. 7 the analog of the present Theorem 5.2 involves the equivalence of the simultaneous unique solvability of two pairs of Marchenko equations to the existence of both a left and a right canonical factorization of a 2×2 matrix function. No such simplification has been found for the present problem.

We conclude this article by giving a sufficient condition for the canonical factorizability of the matrix function in (5.3) and hence of the unique solvability of the solution of the Marchenko equations (4.30)–(4.31).

Corollary 5.3: Suppose $P\in L_1^1(\mathbb{R})$, $Q\in L_2^1(\mathbb{R})$ *and* $W^{\pm}\in L_1^1(\mathbb{R})$, *and let conditions (1)–(3) stated at the beginning of Sec. IV C be fulfilled. Then, for fixed* $x\in\mathbb{R}$, *the system of Marchenko integral equations (4.30) and (4.31) are uniquely solvable if*

$$\sup_{E\in\mathbb{R}}\|\mathbf{F}_l(E,x)\|<1. \tag{5.5}$$

Analogously, for fixed $x \in \mathbb{R}$, the system of Marchenko integral equations (4.32) and (4.33) are uniquely solvable if

$$\sup_{E \in \mathbb{R}} \|\mathbf{F}_r(E,x)\| < 1. \tag{5.6}$$

In (5.5) and (5.6) the norm is defined as the largest singular value of the matrix.

Proof: This corollary is immediate from Theorem 5.2 by observing that (5.5) implies that $\mathbf{F}_l(E,x)$ has a canonical factorization (cf. Ref. 25, Sec. III A). □

ACKNOWLEDGMENTS

One of the authors (V. P.) wishes to express his gratitude to the Department of Mathematics of the University of Cagliari for its hospitality during a visit in which a major part of the research was done.

[1] M. Jaulent and C. Jean, ''The inverse *s*-wave scattering problem for a class of potentials depending on energy,'' Commun. Math. Phys. **28**, 177–220 (1972).
[2] M. Jaulent and C. Jean, ''The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. I,'' Ann. Inst. Henri Poincaré, Sect. A **25**, 105–118 (1976).
[3] M. Jaulent and C. Jean, ''The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. II,'' Ann. Inst. Henri Poincaré, Sect. A **25**, 119–137 (1976).
[4] M. Jaulent, ''Inverse scattering problems in absorbing media,'' J. Math. Phys. **17**, 1351–1360 (1976).
[5] D. H. Sattinger and J. Szmigielski, ''Energy dependent scattering theory,'' Differ. Integral Eqs. **8**, 945–959 (1995).
[6] T. Aktosun, M. Klaus, and C. van der Mee, ''Wave scattering in one dimension with absorption,'' J. Math. Phys. **39**, 1957–1992 (1998).
[7] T. Aktosun, M. Klaus, and C. van der Mee, ''Inverse scattering in one-dimensional nonconservative media,'' Integral Equations and Operator Theory **30**, 297–316 (1998).
[8] D. J. Kaup, ''Finding eigenvalue problems for solving nonlinear evolution equations,'' Prog. Theor. Phys. **54**, 72–78 (1975).
[9] D. J. Kaup, ''A higher-order water-wave equation and the method for solving it,'' Prog. Theor. Phys. **54**, 396–408 (1975).
[10] D. H. Sattinger and J. Szmigielski, ''A Riemann–Hilbert problem for an energy dependent Schrödinger operator,'' Inverse Probl. **12**, 1003–1025 (1996).
[11] E. Corinaldesi, ''Construction of potentials from phase shift and binding energies of relativistic equations,'' Nuovo Cimento **11**, 468–478 (1954).
[12] A. Degasperis, ''On the inverse problem for the Klein–Gordon *s*-wave equation,'' J. Math. Phys. **11**, 551–567 (1970).
[13] R. Weiss and G. Scharf, ''The inverse problem of potential scattering according to the Klein–Gordon equation,'' Helv. Phys. Acta **44**, 910–929 (1971).
[14] V. N. Pivovarchik, ''On the number of eigenvalues of the spectral problem associated with the Klein–Gordon equation,'' in *Schrödinger Operators, Standard and Non-standard*, Proceedings of a Conference held in Dubna, USSR, 6–10 September, 1988; edited by P. Exner and P. Seba (World Scientific, Singapore, 1989), pp. 401–402.
[15] L. D. Faddeev, ''Properties of the *S*-matrix of the one-dimensional Schrödinger equation,'' Am. Math. Soc. Trans. **2**, 139–166 (1964) [Trudy Mat. Inst. Steklova **73**, 314-336 (1964) (in Russian)].
[16] P. Deift and E. Trubowitz, ''Inverse scattering on the line,'' Commun. Pure Appl. Math. **32**, 121–251 (1979).
[17] T. Aktosun and M. Klaus, ''Asymptotics of the scattering coefficients for a generalized Schrödinger equation,'' J. Math. Phys. **40**, 3701–3709 (1999).
[18] R. P. Boas, *Entire Functions* (Academic, New York, 1954).
[19] V. N. Pivovarchik, ''On positive spectra of one class of polynomial operator pencils,'' Integral Equations and Operator Theory **19**, 314–326 (1994).
[20] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon, Oxford, 1937).
[21] J. R. Partington, *An Introduction to Hankel Operators*, London Mathematical Society Student Texts (Cambridge University Press, Cambridge, 1988), Vol. 13.
[22] S. C. Power, *Hankel Operators on Hilbert Space*, Research Notes in Mathematics (Pitman, Boston, 1982), Vol. 64.
[23] I. A. Feldman, ''On an effective solution of certain integral equations on the line and on the half-line,'' Izv. Akad. Nauk Mold. SSR **10**, 16–26 (1961) (in Russian).
[24] I. Feldman, I. Gohberg, and N. Krupnik, ''A method of explicit factorization of matrix functions and applications,'' Integral Equations Operator Theory **18**, 277–302 (1994).
[25] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators* (Birkhäuser, Basel, 1981).