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An inverse problem by eigenvalues of four spectra

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ABSTRACT

Certain parts of the Dirichlet–Dirichlet, Neumann–Dirichlet, Dirichlet–Neumann and Neumann–Neumann spectra are used to find the potential of the Sturm–Liouville equation on a finite interval. This problem possesses a unique solution. Conditions are found necessary and sufficient for four sequences to be the corresponding parts of the four spectra.

1. Introduction

We consider the Sturm–Liouville boundary value problems with Dirichlet and Neumann boundary conditions on a finite interval $[0, a]$. By the Dirichlet–Dirichlet problem we mean the one with the Dirichlet conditions at both ends of the interval (see problem (2.1), (2.2)), by Neumann–Dirichlet the problem with the Neumann condition at the left end and the Dirichlet condition at the right end (see problem (2.1), (2.3)) and so on.

It is well known that the spectra of the Neumann–Dirichlet (or the Dirichlet–Neumann) and the Dirichlet–Dirichlet boundary value problems generated by the same potential uniquely determine this potential in $L_1(0, a)$. Also it is known that the spectra of two boundary value problems with the same Robin boundary condition at one of the ends and different Robin conditions at the other end of the interval uniquely determine the potential and the constants in the boundary conditions. These results are due to Borg [1] (see also [2–4]). If the boundary conditions are given data then the problem of recovering the potential appears to be overdetermined (in the case of Robin conditions). One needs to know not all the eigenvalues of the two spectra. This was shown in [5] and is sometimes called the 'missing eigenvalue problem' (see [6]). Further development of this theory lies in the use of one spectrum together with the knowledge of a part of the potential [7,5,8,6,9].

Another direction of generalization of the above results is to use eigenvalues of more than two spectra to determine the potential. In [10] it was shown that 2/3 part of the union of three spectra of boundary value problems with the same boundary condition at one of the ends uniquely determine the potential. In [11] a similar but more general sufficient condition of unique solvability was given for the case when the known eigenvalues were taken from n different spectra (see [12] for a topical review).

In the present paper we consider real potentials from $L_2(0, a)$ which enables us to use interpolation in the Paley–Wiener class using the results of [13,14]. We use eigenvalues of four boundary value problems, namely the Dirichlet–Dirichlet, the Neumann–Dirichlet, the Dirichlet–Neumann and the Neumann–Neumann problem to recover the potential.

In Section 2 we describe some well known facts about interlacing properties of eigenvalues of the Dirichlet–Dirichlet, the Neumann–Dirichlet, the Dirichlet–Neumann and the Neumann–Neumann problem and the eigenvalue asymptotics

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of these problems. We reformulate known results of [4] in the form of a theorem on solvability and uniqueness of solution of a functional equation. This theorem is used in Section 3 where we prove that certain parts of the spectra of the Dirichlet–Dirichlet, the Neumann–Dirichlet, the Dirichlet–Neumann and the Neumann–Neumann problem uniquely determine the potential. We also characterize the given data of such inverse problem, i.e. we give conditions necessary and sufficient for four sequences of real numbers to be the squares of eigenvalues of certain parts of the spectra of the mentioned problems and describe the procedure of recovering the potential. We use the method that was earlier used in [15] to solve the three spectral inverse problem and in [16,17] to solve the inverse problem on a star graph.

2. Direct problems

Let us consider four boundary value problems with a real potential $q \in L_2(0, a)$: the Dirichlet–Dirichlet problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a] \tag{2.1}$$

$$y(0) = y(a) = 0, \tag{2.2}$$

the spectrum of which we denote by $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\nu_{-k} = -\nu_k)$, the Neumann–Dirichlet problem

$$\begin{aligned} &-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a] \\ &y'(0) = y(a) = 0, \end{aligned} \tag{2.3}$$

with the spectrum denoted by $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\mu_{-k} = -\mu_k)$, the Dirichlet–Neumann problem

$$\begin{aligned} &-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a] \\ &y(0) = y'(a) = 0, \end{aligned} \tag{2.4}$$

with the spectrum which we denote by $\{\kappa_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\kappa_{-k} = -\kappa_k)$, and the Neumann–Neumann problem

$$\begin{aligned} &-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a] \\ &y'(0) = y'(a) = 0, \end{aligned} \tag{2.5}$$

with the spectrum which we denote by $\{\zeta_k\}_{-\infty,\ k\neq 0}^{\infty} \cup \{\zeta_{-0}, \zeta_{+0}\}$ $(\zeta_{-k} = -\zeta_k)$. This way of enumeration appears to be convenient.

Let us denote by $s_j(\lambda, x)$ the solution of the Sturm–Liouville equation (2.1) which satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$ and by $c_j(\lambda, x)$ the solution which satisfies the conditions $c_j(\lambda, 0) - 1 = c_j'(\lambda, 0) = 0$. According to [4]

$$\begin{aligned} s(\lambda, x) &= \frac{\sin \lambda x}{\lambda} + \int_0^x K(x, t) \frac{\sin \lambda t}{\lambda} dt \\ &= \frac{\sin \lambda x}{\lambda} - K(x, x) \frac{\cos \lambda x}{\lambda^2} + \int_0^x K_t(x, t) \frac{\cos \lambda t}{\lambda^2} dt, \end{aligned} \tag{2.6}$$

where

$$K(x, t) = \tilde{K}(x, t) - \tilde{K}(x, -t), \qquad K_t(x, t) = \frac{\partial K(x, t)}{\partial t}$$

and $\tilde{K}(x, t)$ is the solution of the integral equation

$$\tilde{K}(x, t) = \frac{1}{2} \int_0^{\frac{x+t}{2}} q(s) ds + \int_0^{\frac{x+t}{2}} d\alpha \int_0^{\frac{x-t}{2}} q(\alpha + \beta) \tilde{K}(\alpha + \beta, \alpha - \beta) d\beta.$$

The solution $\tilde{K}(x, t)$ possesses partial derivatives of the first order each belonging to $L_2(0, a)$ as a function of each of its variables. Moreover, $K(x, 0) = 0$ and

$$K(x, x) = \frac{1}{2} \int_0^x q(t) dt.$$

It is clear also that

$$s'(\lambda, x) = \cos \lambda a + K(x, x) \frac{\sin \lambda a}{\lambda} + \int_0^x K_x(x, t) \frac{\sin \lambda t}{\lambda}, \tag{2.7}$$

$$\begin{aligned} c(\lambda, x) &= \cos \lambda x + \int_0^x B(x, t) \cos \lambda t dt \\ &= \cos \lambda x + B(x, x) \frac{\sin \lambda x}{\lambda} + \int_0^x B_t(x, t) \frac{\cos \lambda t}{\lambda} dt, \end{aligned} \tag{2.8}$$

$$c'(\lambda, x) = -\lambda \sin \lambda x + B(x, x) \sin \lambda x + \int_0^x B_t(x, t) \cos \lambda t dt, \tag{2.9}$$

$$B(x, t) = \tilde{K}(x, t) + \tilde{K}(x, -t),$$

$$B(x, x) = \frac{1}{2} \int_0^x q(t) dt.$$

Using (2.6)–(2.9) we obtain

$$s(\lambda, a) = \frac{\sin \lambda a}{\lambda} - K \frac{\cos \lambda a}{\lambda^2} + \frac{\psi_1(\lambda)}{\lambda^2},$$

$$c(\lambda, a) = \cos \lambda a + K \frac{\sin \lambda a}{\lambda} + \frac{\psi_2(\lambda)}{\lambda},$$

$$s'(\lambda, a) = \cos \lambda a + K \frac{\sin \lambda a}{\lambda} + \frac{\psi_3(\lambda)}{\lambda},$$

$$c'(\lambda, a) = -\lambda \sin \lambda a + K \cos \lambda a + \psi_4(\lambda),$$

where $K \stackrel{\text{def}}{=} K(a, a)$, $\psi_j \in \mathcal{L}^a$ $(j = 1, 2, 3, 4)$ and $\mathcal{L}^a$ is the Paley–Wiener class of entire functions of exponential type $\leq a$ which belong to $L_2(-\infty, \infty)$ for real λ. Moreover, $\psi_1(0) = K$, $\psi_2(0) = \psi_3(0) = 0$ otherwise $s(\lambda, a)$, $s'(\lambda, a)$ or $c(\lambda, a)$ would have a pole at $\lambda = 0$. By the Paley–Wiener theorem $\mathcal{L}^a$- functions are the Fourier images of all square summable functions supported on $[0, a]$. It is clear that $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ is the set of zeros of $s(\lambda, a)$, $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ is the set of zeros of $c(\lambda, a)$, $\{\kappa_k\}_{-\infty,\ k\neq 0}^{\infty}$ is the set of zeros of $s'(\lambda, a)$ and $\{\zeta_k\}_{-\infty,\ k\neq 0}^{\infty} \cup \{\zeta_{-0}, \zeta_{+0}\}$ is the set of zeros of $c'(\lambda, a)$. Let us mention one more well known result (see, e.g. [4]). The eigenvalues of problems (2.1), (2.2); (2.1), (2.3); (2.1), (2.4) and (2.1), (2.5) behave asymptotically as follows:

$$\nu_k \underset{k\to+\infty}{=} \frac{\pi k}{a} + \frac{K}{\pi k} + \frac{\alpha_k^{(1)}}{k}, \tag{2.10}$$

$$\mu_k \underset{k\to+\infty}{=} \frac{\pi(k - 1/2)}{a} + \frac{K}{\pi k} + \frac{\alpha_k^{(2)}}{k}, \tag{2.11}$$

$$\kappa_k \underset{k\to+\infty}{=} \frac{\pi(k - 1/2)}{a} + \frac{K}{\pi k} + \frac{\alpha_k^{(3)}}{k}, \tag{2.12}$$

$$\zeta_k \underset{k\to+\infty}{=} \frac{\pi k}{a} + \frac{K}{\pi k} + \frac{\alpha_k^{(4)}}{k}, \tag{2.13}$$

where $\{\alpha_k^{(j)}\}_{k=1}^{\infty} \in l_2$ for $j = 1, 2, 3, 4$. Also it is known that

$$-\infty < \mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \cdots \tag{2.14}$$

$$-\infty < \kappa_1^2 < \nu_1^2 < \kappa_2^2 < \nu_2^2 < \cdots \tag{2.15}$$

$$-\infty < \zeta_{+0}^2 < \mu_1^2 < \zeta_1^2 < \mu_2^2 < \cdots \tag{2.16}$$

and

$$-\infty < \zeta_{+0}^2 < \kappa_1^2 < \zeta_1^2 < \kappa_2^2 < \cdots. \tag{2.17}$$

The following theorem is a reformulation of Theorem 3.4.1 in [4].

Theorem 2.1 ([4]). *For two sequences $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ and $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ of numbers to be the spectra of Neumann–Dirichlet and Dirichlet–Dirichlet problems generated by the same real potential $q \in L_2(0, a)$ it is necessary and sufficient that the following conditions should be satisfied.*

1. $\mu_{-k} = -\mu_k$, $\nu_{-k} = -\nu_k$.
2. *The sequences are interlaced as in* (2.14).
3. *Asymptotics* (2.10) *and* (2.11) *are true.*

The following theorem is a direct consequence of Theorem 2.1.

Theorem 2.2. *Let f_1 and g_1 be real entire functions of exponential type a of the form*

$$g_1(\lambda) = \frac{\sin \lambda a}{\lambda} - K \frac{\cos \lambda a}{\lambda^2} + \frac{\tilde{\psi}_1(\lambda)}{\lambda^2}, \tag{2.18}$$

$$f_1(\lambda) = \cos \lambda a + K \frac{\sin \lambda a}{\lambda} + \frac{\tilde{\psi}_2(\lambda)}{\lambda}, \tag{2.19}$$

where K is a constant, and ψ_1 and ψ_2 belong to $\mathcal{L}^a$. Let the zeros $\{a_k\}_{-\infty,\ k\neq 0}^{\infty}$ of f_1 be interlaced with the zeros of $\{b_k\}_{-\infty,\ k\neq 0}^{\infty}$ of g_1 in the following way:

$$-\infty < a_1^2 < b_1^2 < a_2^2 < b_2^2 < \cdots \quad (a_{-k} = -a_k,\ b_{-k} = -b_k).$$

Then equation

$$f_1(\lambda)g_2(\lambda) - f_2(\lambda)g_1(\lambda) = 1 \tag{2.20}$$

possesses a unique solution (f_2, g_2) in the class of pairs of real entire functions of exponential type a which satisfy the condition

$$g_2(\lambda) = \cos\lambda a + K\frac{\sin\lambda a}{\lambda} + \frac{\tilde{\psi}_3(\lambda)}{\lambda}, \tag{2.21}$$

where $\tilde{\psi}_3 \in \mathcal{L}^a$.

Proof. According to [4, Lemma 3.4.2] Eqs. (2.18) and (2.19) imply

$$b_k \underset{k\to+\infty}{=} \frac{\pi k}{a} + \frac{K}{\pi k} + \frac{\beta_k}{k}$$

where $\{\beta_k\}_{k=1}^{\infty} \in l_2$,

$$a_k \underset{k\to+\infty}{=} \frac{\pi(k-1/2)}{a} + \frac{K}{\pi k} + \frac{\alpha_k}{k}$$

where $\{\alpha_k\}_{k=1}^{\infty} \in l_2$ This means the sequences $\{a_k\}_{-\infty,\ k\neq 0}^{\infty}$ and $\{b_k\}_{-\infty,\ k\neq 0}^{\infty}$ satisfy conditions of Theorem 2.1 and, therefore, there exists a real function $q \in L_2(0, a)$ for which $\{b_k\}_{-\infty,\ k\neq 0}^{\infty}$ is the spectrum of the Dirichlet–Dirichlet problem and $\{a_k\}_{-\infty,\ k\neq 0}^{\infty}$ is the spectrum of the Neumann–Dirichlet problem. The functions g_1 and f_1 are the corresponding characteristic functions of these two problems, i.e. $g_1(\lambda) = s(\lambda, a)$ and $f_1(\lambda) = c(\lambda, a)$. If we solve the Dirichlet–Neumann and Neumann–Neumann problems with the obtained potential then the characteristic functions of these problems are $s'(\lambda, a)$ and $c'(\lambda, a)$. The Lagrange identity is

$$c(\lambda, a)s'(\lambda, a) - s(\lambda, a)c'(\lambda, a) = 1 = f_1(\lambda)s'(\lambda, a) - g_1(\lambda)c'(\lambda, a).$$

This means that there exists a solution to (2.20). Let us show that it is unique. Suppose there exists another solution $(u, v) \not\equiv (s'(\lambda, a), c'(\lambda, a))$ to (2.20) and such that

$$u = \cos\lambda a + K\frac{\sin\lambda a}{\lambda} + \frac{\tilde{\psi}_3(\lambda)}{\lambda},$$

where $\tilde{\psi}_3 \in \mathcal{L}^a$. Then

$$f_1(\lambda)(s'(\lambda, a) - u(\lambda)) - g_1(\lambda)(c'(\lambda, a) - v(\lambda)) = 0$$

or if $(c'(\lambda, a) - v(\lambda)) \not\equiv 0$

$$\frac{g_1(\lambda)}{f_1(\lambda)} = \frac{\tilde{\psi}_4(\lambda)}{\lambda(c'(\lambda, a) - v(\lambda))}$$

where $\tilde{\psi}_4 \in \mathcal{L}^a$. The last equation is false because g_1 is a sine-type function while ψ_3 is not.

Let us recall that a function f is said to be of sine-type (see [13]), if its zeros are all distinct and there exist positive numbers m, M and p such that

$$me^{|\mathrm{Im}\,\lambda| a} \le |f(\lambda)| \le Me^{|\mathrm{Im}\,\lambda| a}$$

for $|\mathrm{Im}\,\lambda| > p$.

Thus, $(c'(\lambda, a) - v(\lambda)) \equiv 0$ and, consequently, $s'(\lambda, a) - u(\lambda) \equiv 0$. Theorem is proved. □

3. Inverse problems

Theorem 3.1. *Let $\{k_j\}_{j\in A_1\subset\mathbb{N}}$, $\{p_j\}_{j\in A_2\subset\mathbb{N}}$, $\{r_j\}_{j\in A_3\subset\mathbb{N}}$ and $\{s_j\}_{j\in A_4\subset\mathbb{N}}$ be sequences of natural numbers such that $\{k_j\}_{j\in A_1\subset\mathbb{N}} \cap \{p_j\}_{j\in A_2\subset\mathbb{N}} = \emptyset$, $\{k_j\}_{j\in A_1\subset\mathbb{N}} \cup \{p_j\}_{j\in A_2\subset\mathbb{N}} = \mathbb{N}$, $\{r_j\}_{j\in A_3\subset\mathbb{N}} \cap \{s_j\}_{j\in A_4\subset\mathbb{N}} = \emptyset$, $\{r_j\}_{j\in A_3\subset\mathbb{N}} \cup \{s_j\}_{j\in A_4\subset\mathbb{N}} = \mathbb{N}$.*

Let four sequences of real numbers $\{\nu_{k_j}^2\}$, $\{\zeta_{p_j}^2\}$, $\{\mu_{r_j}^2\}$ and $\{\kappa_{s_j}^2\}$ be given unions of which $\{\xi_k^2\}_{k=1}^{\infty} = \{\nu_{k_j}^2\} \cup \{\zeta_{p_j}^2\}$ and $\{\upsilon_k^2\}_{k=1}^{\infty} = \{\mu_{r_j}^2\} \cup \{\kappa_{s_j}^2\}$ being reenumerated monotonically are interlaced:

$$-\infty < \upsilon_1^2 < \xi_1^2 < \upsilon_2^2 < \xi_2^2 < \cdots$$

and behave asymptotically as follows:

$$\xi_k \underset{k\to+\infty}{=} \frac{\pi k}{a} + \frac{A}{\pi k} + \frac{\beta_k^{(1)}}{k}, \quad (\xi_{-k} = -\xi_k), \tag{3.1}$$

$$\upsilon_k \underset{k\to+\infty}{=} \frac{\pi\left(k-\frac{1}{2}\right)}{a} + \frac{A}{\pi k} + \frac{\beta_k^{(2)}}{k}, \quad (\upsilon_{-k} = -\upsilon_k), \tag{3.2}$$

where A is a real constant, $\{\beta_k^{(j)}\}_{k=1}^{\infty} \in l_2$ for $j = 1, 2$. Then there exists a unique real potential $q \in L_2(0,a)$ such that $\{\nu_{k_j}\}$ $(\nu_{-k_j} = -\nu_{k_j})$ are eigenvalues of problem (2.1), (2.2), $\{\mu_{r_j}\}$ $(\mu_{-r_j} = -\mu_{r_j})$ are eigenvalues of problem (2.1), (2.3), $\{\kappa_{s_j}\}$ $(\kappa_{-s_j} = -\zeta_{s_j})$ are eigenvalues of problem (2.1), (2.4) and $\{\zeta_{p_j}^2\}$ $(\zeta_{-p_j} = -\zeta_{p_j})$ are eigenvalues of problem (2.1), (2.5).

Proof. Let us notice that the sequences $\{\xi_k\}_{-\infty,\,k\neq0}^{\infty}$ and $\{\upsilon_k\}_{-\infty,\,k\neq0}^{\infty}$ satisfy conditions of Theorem 2.1. Therefore, there exists a real potential $\hat{q} \in L_2(0,a)$ for which $\{\xi_k\}_{-\infty,\,k\neq0}^{\infty}$ is the Dirichlet–Dirichlet spectrum and $\{\upsilon_k\}_{-\infty,\,k\neq0}^{\infty}$ is the Neumann–Dirichlet spectrum. We do not need to construct this potential but we need to find the Dirichlet–Neumann and Neumann–Neumann spectra generated by $\hat{q}$.

For the sake of simplicity let us assume that $0 \notin \{\xi_k^2\}_{k=1}^{\infty} \cup \{\upsilon_k^2\}_{k=1}^{\infty}$. Otherwise, we can shift the spectral parameter $\lambda^2 \to \lambda^2 + c$.

Using $\{\upsilon_k\}_{-\infty,k\neq0}^{\infty}$ we construct

$$\hat{\phi}(\lambda) = \prod_{k=1}^{\infty}\left(\frac{a}{\pi\left(k-\frac{1}{2}\right)}\right)^2 (\upsilon_k^2 - \lambda^2).$$

Then according to [4, Lemma 3.4.2] we have

$$\hat{\phi}(\lambda) = \cos\lambda a + A\frac{\sin\lambda a}{\lambda} + \frac{\hat{\tau}_2(\lambda)}{\lambda}, \tag{3.3}$$

where $\hat{\tau}_2 \in \mathcal{L}^a$ and the constant A can be found as

$$A = \frac{2\pi}{a}\lim_{p\to+\infty}\left(p\,\hat{\phi}\left(\frac{\pi\left(2p+\frac{1}{2}\right)}{a}\right)\right).$$

The function

$$\lambda\hat{\omega}(\lambda) \overset{\text{def}}{=} \lambda a \prod_{k=1}^{\infty}\left(\frac{a}{\pi k}\right)^2 \quad (\xi_k^2 - \lambda^2)$$

is sine-type. To prove it we notice that according to Lemma 3.4.2 in [4] it is of the form

$$\hat{\omega}(\lambda) = \frac{\sin\lambda a}{\lambda} - A\frac{\cos\lambda a}{\lambda^2} + \frac{\hat{\tau}_1(\lambda)}{\lambda^2}, \tag{3.4}$$

where $\hat{\tau}_1 \in \mathcal{L}^a$. Then taking into account that $\hat{\omega}(0) \neq 0$ we conclude that $\lambda\hat{\omega}(\lambda)$ is a sine-type function.

We choose $\{\xi_k\}_{-\infty,k\neq0}^{\infty} \cup \{0\}$ as the nodes of interpolation for finding a Paley–Wiener function $\hat{\tau}_3(\lambda)$ and as the values at the nodes we choose

$$\hat{\tau}_3(\xi_k) = \xi_k\left(\hat{\phi}^{-1}(\xi_k) - \cos\xi_k a - A\frac{\sin\xi_k a}{\xi_k}\right)$$

for all $k \neq 0$ and we set $\hat{\tau}_3(0) = 0$ for $\xi_0 \overset{\text{def}}{=} 0$. Using (3.3) we obtain

$$\hat{\tau}_3(\xi_k) = \xi_k\left(\frac{1}{\cos\xi_k a + A\frac{\sin\xi_k a}{\xi_k} + \frac{\hat{\tau}_2(\xi_k)}{\xi_k}} - \cos\xi_k a - A\frac{\sin\xi_k a}{\xi_k}\right). \tag{3.5}$$

To estimate asymptotics of $\hat{\tau}_3(\xi_k)$ we notice that (see Lemma 1.4.3 in [4])

$$\{\hat{\tau}_2(\xi_k)\}_{k=1}^{\infty} \in l_2. \tag{3.6}$$

Using (3.1) we arrive at

$$\sin \xi_k a = (-1)^k \frac{Aa}{\pi k} + \frac{\delta_k^{(1)}}{k}, \tag{3.7}$$

$$\cos \xi_k a = (-1)^k + \frac{\delta_k^{(2)}}{k}, \tag{3.8}$$

where $\left\{\delta_k^{(j)}\right\}_{k=-\infty,\ k\neq 0}^{\infty}$ belong to l_2 for $j = 1, 2$.

Using (3.6)–(3.8) we obtain from (3.5):

$$\{\hat{\tau}_3(\xi_k)\}_{-\infty,k\neq 0}^{\infty} \in l_2. \tag{3.9}$$

Therefore, taking into account (3.9) we use Theorem A in [13] (see also [14]) and find

$$\hat{\tau}_3(\lambda) = \lambda \hat{\omega}(\lambda) \sum_{k=-\infty}^{k=+\infty} \frac{\hat{\tau}_3(\xi_k)}{\left.\frac{d\lambda\hat{\omega}(\lambda)}{d\lambda}\right|_{\lambda=\xi_k} (\lambda - \xi_k)}. \tag{3.10}$$

The series on the right hand side of (3.10) converges uniformly on any compact subdomain of $\mathbb{C}$ and in the norm of $L_2(-\infty, +\infty)$ for real λ to a function which belongs to $\mathcal{L}^a$.

Let us notice that according to [13] the obtained $\hat{\tau}_3(\lambda)$ is the unique solution of the following interpolation problem: given the nodes $\{\xi_k\}_{-\infty}^{\infty}$ and the values $\{\hat{\tau}_3(\xi_k)\}_{-\infty}^{\infty}$ at these nodes, find $\hat{\tau}_3$.

Now, we can construct the function

$$\hat{\Phi}(\lambda) = \cos \lambda a + A \frac{\sin \lambda a}{\lambda} + \frac{\hat{\tau}_3(\lambda)}{\lambda}, \tag{3.11}$$

which pretends to be the characteristic function to problem (2.1), (2.4). Here $\int_0^a \hat{q}(x)dx = A$.

Let $\hat{s}(\lambda, x)$ be the solution of Eq. (2.1) with the potential $\hat{q}$ which satisfies conditions $\hat{s}(\lambda, 0) = \hat{s}'(\lambda, 0) - 1 = 0$ and let $\hat{s}'(\lambda, a)$ be the value of its derivative at $x = a$. It is clear that

$$\hat{s}(\lambda, a) \equiv \hat{\omega}(\lambda), \tag{3.12}$$

$$\hat{s}(\xi_k, a) = 0 \tag{3.13}$$

and

$$\hat{s}'(\lambda, a) = \cos \lambda a + A \frac{\sin \lambda a}{\lambda} + \frac{\tilde{\tau}_3(\lambda)}{\lambda}, \tag{3.14}$$

where $\tilde{\tau}_3 \in \mathcal{L}^a$.

Let $\hat{c}(\lambda, x)$ be the solution of Eq. (2.1) with the potential $\hat{q}$ which satisfies conditions $\hat{c}(\lambda, 0) - 1 = \hat{c}'(\lambda, 0) = 0$, Then $\hat{c}(\lambda, a) = \hat{\phi}(\lambda)$ and due to (3.12) and the Lagrange identity we have

$$\hat{c}(\lambda, a)\hat{s}'(\lambda, a) - \hat{s}(\lambda, a)\hat{c}'(\lambda, a) = \hat{\phi}(\lambda)\hat{s}'(\lambda, a) - \hat{\omega}(\lambda)\hat{c}'(\lambda, a) = 1. \tag{3.15}$$

Using (3.13) we obtain from (3.15)

$$\hat{\phi}(\xi_k)\hat{s}'(\xi_k, a) = 1 \tag{3.16}$$

and due to (3.16)

$$\tilde{\tau}_3(\xi_k) = \xi_k \left(\phi^{-1}(\xi_k) - \cos \xi_k a - A \frac{\sin \xi_k a}{\xi_k} \right),$$

for all $k \neq 0$ and $\tilde{\tau}_3(0) = 0$ since $\hat{s}'(\lambda, a)$ is an entire function.

Thus, $\tilde{\tau}_3$ is a solution of the same interpolation problem as τ_3, the problem having a unique solution (see Theorem A in [13]). We conclude that $\tilde{\tau}_3 \equiv \hat{\tau}_3$ and, therefore

$$\hat{\Phi}(\lambda) \equiv \hat{s}'(\lambda, a) \tag{3.17}$$

where $\hat{s}'(\lambda, a)$ is the characteristic function of the Dirichlet–Neumann problem with the potential $\hat{q}$. We denote by $\{\eta_k\}_{-\infty,\ k\neq 0}^{\infty}$ the zeros of $\hat{s}'(\lambda, a)$. Then

$$\hat{\Phi}(\lambda) = \prod_{k=1}^{\infty} \left(\frac{a}{\pi\left(k - \frac{1}{2}\right)} \right)^2 (\eta_k^2 - \lambda^2).$$

Then due to (2.13)

$$\eta_k \underset{k\to+\infty}{=} \frac{\pi\left(k-\frac{1}{2}\right)}{a} + \frac{A}{\pi k} + \frac{\gamma_k^{(1)}}{k}, \tag{3.18}$$

where $\{\gamma_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty} \in l_2$. Now (3.15) implies

$$\hat{F}(\lambda) \overset{\text{def}}{=} \frac{\hat{\phi}(\lambda)\hat{\Phi}(\lambda)-1}{\hat{\omega}(\lambda)} \equiv \hat{c}'(\lambda, a), \tag{3.19}$$

and, consequently,

$$\hat{F}(\lambda) = -\lambda \sin \lambda a + A \cos \lambda a + \hat{\tau}_4(\lambda) \tag{3.20}$$

with $\hat{\tau}_4 \in \mathcal{L}^a$.

Denote by $\{\epsilon_k\}_{-\infty,\ k\neq 0}^{\infty} \cup \{\epsilon_{+0}, \epsilon_{-0}\}$ ($\epsilon_{-k} = -\epsilon_k$ for $k = +0, 1, 2, 3, \ldots$) the zeros of $\hat{c}'(\lambda, a)$. Then due to (2.14) we have

$$\epsilon_k \underset{k\to+\infty}{=} \frac{\pi k}{a} + \frac{A}{\pi k} + \frac{\gamma_k^{(2)}}{k}, \tag{3.21}$$

where $\{\gamma_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty} \in l_2$ and due to (2.14)–(2.17):

$$-\infty < \epsilon_{+0}^2 < \min\{\upsilon_1^2, \eta_1^2\} \le \max\{\upsilon_1^2, \eta_1^2\} < \min\{\epsilon_1^2, \xi_1^2\} \le \max\{\epsilon_1^2, \xi_1^2\} < \min\{\upsilon_2^2, \eta_2^2\} \le \max\{\upsilon_2^2, \eta_2^2\} < \cdots.$$

Since each of the intervals $(\max\{\upsilon_k^2, \eta_k^2\}, \min\{\upsilon_{k+1}^2, \eta_{k+1}^2\})$ ($k = 1, 2, \ldots$) contains exactly one element of the sequence $\{\xi_k^2\}_{k=1}^{\infty}$ and exactly one element of the sequence $\{\epsilon_k^2\}_{k=1}^{\infty}$ while the interval $(-\infty, \min\{\upsilon_1^2, \eta_1^2\})$ contains only ϵ_{+0}^2 we can identify the elements of these sequences as follows:

$$\epsilon_k \overset{\text{def}}{=} \nu_k \quad \text{for } k \in \{p_j\}\ (\nu_{-k} = -\nu_k),$$
$$\epsilon_k \overset{\text{def}}{=} \zeta_k \quad \text{for } k \in \{k_j\} \cup \{0\}\ (\zeta_{-k} = -\zeta_k).$$

Each of the intervals $(\max\{\epsilon_k^2, \xi_k^2\}, \min\{\epsilon_{k+1}^2, \xi_{k+1}^2\})$ ($k = 1, 2, \ldots$) as well as the interval $(\epsilon_{+0}^2, \min\{\epsilon_1^2, \xi_1^2\})$ contains exactly one element of the sequence $\{\upsilon_k^2\}_{k=1}^{\infty}$ and exactly one element of the sequence $\{\eta_k^2\}_{k=1}^{\infty}$. Thus we can identify the elements of these sequences as follows:

$$\eta_k \overset{\text{def}}{=} \mu_k \quad \text{for } k \in \{s_j\}\ (\mu_{-k} = -\mu_k),$$
$$\eta_k \overset{\text{def}}{=} \kappa_k \quad \text{for } k \in \{r_j\}\ (\kappa_{-k} = -\kappa_k).$$

Thus, we have two sequences $\{\nu_k^2\}_{k=1}^{\infty}$ and $\{\mu_k^2\}_{k=1}^{\infty}$ which satisfy the condition

$$-\infty < \mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \cdots.$$

Due to (3.1) and (3.21) the set $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ satisfies the condition

$$\nu_k \underset{k\to+\infty}{=} \frac{\pi k}{a} + \frac{A}{\pi k} + \frac{\sigma_k^{(1)}}{k},$$

while due to (3.2) and (3.18) $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ satisfies

$$\mu_k \underset{k\to+\infty}{=} \frac{\pi\left(k-\frac{1}{2}\right)}{a} + \frac{A}{\pi k} + \frac{\sigma_k^{(2)}}{k},$$

where $\{\sigma_k^{(j)}\}_{-\infty,\ k\neq 0}^{\infty} \in l_2$ for $j = 1, 2$.

Thus, the sets $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ and $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ satisfy the conditions of Theorem 2.1 and, therefore, there exists a unique real function $q(x) \in L_2(0, a)$ which generates Dirichlet–Dirichlet and Neumann–Dirichlet problems on $[0, a]$ with the spectra $\{\nu_k\}_{-\infty,\ k\neq 0}^{\infty}$ and $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$, respectively.

We can find q via procedure [4] described below. Without loss of generality let us assume that $\mu_1^2 > 0$, otherwise we apply a shift of the spectral parameter. The function

$$e(\lambda) = (\phi(\lambda) + i\lambda\omega(\lambda))e^{-i\lambda a}$$

where

$$\lambda\omega(\lambda) \stackrel{\text{def}}{=} \lambda a \prod_{k=1}^{\infty} \left(\frac{a}{\pi k}\right)^2 (\nu_k^2 - \lambda^2), \tag{3.22}$$

$$\phi(\lambda) = \prod_{k=1}^{\infty} \left(\frac{a}{\pi\left(k - \frac{1}{2}\right)}\right)^2 (\mu_k^2 - \lambda^2), \tag{3.23}$$

is the Jost-function of the corresponding prolonged Sturm–Liouville problem on the semi-axis:

$$-y'' + Q(x)y = \lambda^2 y, \quad x \in [0, \infty),$$
$$y(0) = 0$$

with

$$Q(x) = \begin{cases} q(x) & \text{for } x \in [0, a] \\ 0 & \text{for } x \in (a, \infty). \end{cases}$$

Then we construct the S-function of the problem on the semi-axis:

$$S(\lambda) = \frac{e(\lambda)}{e(-\lambda)}$$

and the function

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S(\lambda)) e^{i\lambda x} dx.$$

Solving the Marchenko equation

$$K(x, t) + G(x + t) + \int_x^{\infty} K(x, s) G(s + t) ds = 0$$

we find $K(x, t)$ and the potential:

$$q(x) = 2\frac{dK(x, x)}{dx}, \quad x \in [0, a]$$

which is a real function and belongs to $L_2(0, a)$. If $s(\lambda, x)$ and $c(\lambda, x)$ are the corresponding solutions of (2.1) with the potential $q(x)$ then $c(\lambda, a) = \phi(\lambda)$ and $s(\lambda, a) = \omega(\lambda)$. It means that $s(\nu_k, a) = 0$ for all $k \in \{k_j\}$ and $c(\mu_k, a) = 0$ for all $k \in \{r_j\}$. It remains to prove that $s'(\kappa_k, a) = 0$ for all $k \in \{s_j\}$ and $c'(\zeta_k, a) = 0$ for all $k \in \{p_j\}$.

Denote

$$F(\lambda) = a(\zeta_{+0}^2 - \lambda^2) \prod_{k=1}^{\infty} \left(\frac{a}{\pi k}\right)^2 (\zeta_k^2 - \lambda^2).$$

It is clear that

$$F(\lambda) = -\lambda \sin \lambda a + A \cos \lambda a + \tau_4(\lambda), \tag{3.24}$$

where $\tau_4 \in \mathcal{L}^a$. We also consider the function

$$\Phi(\lambda) \stackrel{\text{def}}{=} \prod_{k=1}^{\infty} \left(\frac{a}{\pi\left(k - \frac{1}{2}\right)}\right)^2 (\kappa_k^2 - \lambda^2)$$

which admits the representation

$$\Phi(\lambda) = \cos \lambda a + A\frac{\sin \lambda a}{\lambda} + \frac{\tau_3(\lambda)}{\lambda} \tag{3.25}$$

where $\tau_3 \in \mathcal{L}^a$.

Setting $\zeta_{+0} = -\zeta_{-0} \stackrel{\text{def}}{=} \epsilon_{+0} = -\epsilon_{-0}$ we see that by the definitions $\{\nu_k^2\}_{k=1}^{\infty} \cup \{\zeta_k^2\}_{k=+0}^{\infty} = \{\xi_k^2\}_{k=1}^{\infty} \cup \{\epsilon_k^2\}_{k=+0}^{\infty}$. That means that the sets of zeros of the functions $\omega(\lambda)F(\lambda)$ and $\hat{\omega}(\lambda)\hat{F}(\lambda)$ coincide. Using (3.4), (3.20), (3.24) and representation

$$\omega(\lambda) = \frac{\sin \lambda a}{\lambda} + A\frac{\cos \lambda a}{\lambda^2} + \frac{\tau(\lambda)}{\lambda^2}$$

which follows from (3.22) we obtain

$$\omega(\lambda)F(\lambda) \equiv \tilde{\omega}(\lambda)\tilde{F}(\lambda). \tag{3.26}$$

By the definitions $\{\mu_k^2\}_{k=1}^{\infty} \cup \{\kappa_k^2\}_{k=1}^{\infty} = \{\upsilon_k^2\}_{k=1}^{\infty} \cup \{\eta_k^2\}_{k=1}^{\infty}$. Thus the sets of zeros of the functions $\hat{\phi}(\lambda)\hat{\Phi}(\lambda)$ and $\phi(\lambda)\Phi(\lambda)$ coincide. Using (3.3), (3.11), (3.25) and the representation

$$\phi(\lambda) = \cos\lambda a + A\frac{\sin\lambda a}{\lambda} + \frac{\tau_2(\lambda)}{\lambda},$$

where $\tau_2 \in \mathcal{L}^a$ which follows from (3.23) we conclude that

$$\Phi(\lambda)\phi(\lambda) \equiv \tilde{\Phi}(\lambda)\tilde{\phi}(\lambda). \tag{3.27}$$

Substituting (3.17) and (3.19) into (3.15) and using (3.26) and (3.27) we obtain

$$\phi(\lambda)\Phi(\lambda) - \omega(\lambda)F(\lambda) = 1.$$

On the other hand, the Lagrange identity is

$$\phi(\lambda)s'(\lambda, a) - \omega(\lambda)c'(\lambda, a) = 1.$$

According to Theorem 2.2 the equation

$$\phi(\lambda)u(\lambda) - \omega(\lambda)v(\lambda) = 1$$

possesses a unique solution and therefore $F(\lambda) \equiv c'(\lambda, a)$ and $\Phi(\lambda) = s'(\lambda, a)$. Therefore, $c'(\zeta_k, a) = 0$ for all $k \in \{p_j\}$ and $s'(\kappa_k, a)$ for all $k \in \{s_j\}$.

Uniqueness of the solution of our inverse problem follows from uniqueness of the choice of μ_k for $k \in \{s_j\}$ and ν_k for $k \in \{p_j\}$ and uniqueness of the potential corresponding to $\{\mu_k\}_{k=1}^{\infty}$ and $\{\nu_k\}_{k=1}^{\infty}$. Theorem is proved. □

Remark. According to (2.10)–(2.17) the conditions of Theorem 3.1 are necessary and sufficient.

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