MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
State institution:
"SOUTH UKRAINIAN NATIONAL PEDAGOGICAL UNIVERSITY NAMED AFTER K.D. USHINSKY»

Discrete Mathematics
Lecture Notes

Odesa 2025

УДК 519.101

Recommended for publication by the Academic Council of the State Institution "South Ukrainian National Pedagogical University named after K.D. Ushinsky", record #12, date 27/03/2025.

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The lecture notes in English are intended for students of higher education institutions, in particular for students of the Educational and Scientific Institute of Natural and Mathematical Sciences, Informatics and Management of Ushinsky University, who study the academic discipline "Discrete Mathematics". The lecture notes consider such a section of discrete mathematics as combinatorics.

CONTENTS

Introduction.

Discrete mathematics historically arose when mankind used numerical systems in ancient times (numbers written in non-positional, and later in positional numerical systems) and the associated algorithms for performing arithmetic operations, solving equations, etc. Discrete mathematics began to stand out as a separate section of mathematics in the 17th century, which is associated with the works of Leonard Euler in the field of combinatorial analysis and graph theory, and Jacob Bernoulli in the field of combinatorial probability theory. A major role in the development of discrete mathematics was played by G. W. Leibniz.

In the 19th century, famous mathematicians J. L. Lagrange, A. Cayley, J. Boole, C. Jordan and many others worked in the field of discrete mathematics. In the 20th century, a significant influence on the development of discrete mathematics was exerted by the research ~~and~~ work of A. Poincaré and D. Hilbert, E. L. Post, A. M. Turing and others. The rapid development of discrete mathematics in the second half of the 20th century is associated with the "digital revolution" in telecommunications and computing. Discrete mathematics became the basis for the design and application of numerous digital electronic devices. The first applications of discrete mathematics in this field are associated with the names of V. A. Kotelnikov, K. E. Shannon.

The name of the discipline "Discrete Mathematics" characterizes the constructive nature of the discipline, which studies algorithmic and combinatorial methods of mathematics. Discrete mathematics was separated into an independent discipline in connection with the emergence and development of computers and computer technologies, although, for example, combinatorics is much older than some sections of mathematics. Discrete mathematics problems are usually closely related to computer science, computing, and technical mathematics programs and are expressed in the form of various algorithms. The need to study proof techniques for future mathematicians is obvious. It is also very important for the development of logical thinking in future specialists in the field of computer science.

Typical examples of applications of various sections of discrete mathematics are methods of mathematical modeling, logistics, cryptographic protocols, information coding theory, computer algorithms, algorithm complexity theory, software testing, etc.

Discrete (or finite) mathematics (sometimes discrete analysis) studies finite sets and various structures on finite sets that arise in various sections of mathematics. This means that the concepts of the limit of a sequence or function, the continuity of a function are not the subject of study of this section of mathematics, although they can be used as auxiliary tools.

The lecture notes are based on the course of lectures "Discrete Mathematics", which the authors gave for mathematics students of the Educational and Scientific Institute of Natural and Mathematical Sciences, Informatics and Management in English.

The lecture notes consider the problems and algorithms of combinatorics.

The basics of combinatorics are presented, both classical combinatorics problems (the problem of counting lottery numbers, the number of ways to divide a group into subgroups) and the problem of dividing sets, and the problem of calculating injective or surjective functions acting on given sets are considered.

COMBINATORICS

§ 1. Basic rules (principles) of combinatorics

Let us denote by | | the cardinality of a set (for a finite set, this is the number of elements).

Let A and B be finite sets with cardinalities $|A| = n, |B| = m$.

Combinatorics is based on the following two rules (of sum and of product):

1. The addition principle

1. $|A \cup B| = |A| + |B|$, if $A \cap B = \emptyset$.

This follows from the inclusion-exclusion formula. Indeed, the element $a \in A$ can be chosen in n ways, the element $b \in B$ can be chosen in m ways. Then the element $x \in A \cup B$ can be chosen in $n + m$ ways.

Example 1. There are 3 apples and 4 pears on a plate. In how many ways can one fruit be chosen from the plate?

Since $|A| = 3, |P| = 4, A \cap P = \emptyset$, therefore $|A \cup P| = 3 + 4 = 7$ ways.

Remark. The formula can be generalized to a larger number of sets. Let $A_1, A_2, \cdots, A_k$ be pairwise distinct sets (that is $A_i \cap A_j = \emptyset$, if $i \neq j$) . Then the number of elements in their union, that is, the number of ways to choose one of them, is equal to

$$|A_1 \cup \ldots \cup A_k| = |A_1| + \cdots + |A_k| = \sum_{i=1}^{k} |A_i|.$$

2. Multiplication Principle

Let us recall the definition of the Cartesian product of two sets. The Cartesian (direct) product of sets A and B is called the set of ordered pairs of the form (a, b), where $a \in A, b \in B$.

$$A \times B = \{(a, b), a \in A, b \in B\}.$$

Multiplication Principle (Direct product rule):

$$|A \times B| = |A| \cdot |B|.$$

Indeed, let the sets $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_m\}$. The choice of an element from the set *A* does not depend on the choice of an element from the set *B*.

Let us count how many ordered pairs of the form $(a_i, b_j), i = \overline{1,n},\ j = \overline{1,m},$ can be made. Let's fix the element $a_1 \in A$. You can pair this item as follows $(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_m),$ totally m pairs. Similarly, with the element a_2, m pairs can be formed with the element a_2, also with each element a_i . Thus, all in all $n \cdot m$ pairs can be made. That is, $|A \times B| = n \cdot m$.

Example 2. In the school cafeteria, a starter can be chosen out of 2, and a main course out of 3. How many lunch options can be made of a starter and a main course?

According to the Multiplication Principle, the number of options for choosing courses must be multiplied: 2·3=6 .

Remark. The formula can be generalized to a larger number of sets.

Example 3. How many three-digit numbers are there that are divisible by 5?

A three-digit number can be written in the form $\overline{abc}$, where the digit $c \in \{0; 5\} := \mathrm{C}$, because according to the condition of the problem, the number is divisible by 5, that is, it ends with the digit 0 or 5, therefore $|C|$=2. The digit $b \in \{0, \ldots, 9\} := B$, $|B|$=10, the digit $a \in \{1, \ldots, 9\} = A$, $|A|$=9.

$$|A \times B \times C| = |A| \cdot |B| \cdot |C| = 9 \cdot 10 \cdot 2 = 180 \text{ numbers.}$$

That is, the first digit can be chosen in 9 ways, the second in 10 ways, and the third in 2 ways. According to the product rule, you need to multiply the numbers of ways to choose the total amount of numbers.

Example 4. Find the number of four-digit numbers.

A four-digit number is written in the form $\overline{abcd}$, where the digit $a \in \{1, \ldots, 9\}$, digits $b, c, d \in \{0, \ldots, 9\}$. Then the number of four-digit numbers is $9 \cdot 10^3$=9000.

Example 5. In the school cafeteria, a starter can be chosen out of 3, a main course out of 6, and a desert out of 5. How many lunch options can be made of a starter and a main course or a main course and a desert?

Let us denote by A_1 the number of ways to choose a starter and a main course, and by A_2 the number of ways to choose a main course and a desert. Then we find the desired number using the formula:

$|A_1 \cup A_2| = |A_1| + |A_2| =$ (by the addition principle) = 3·6+ 6·5=48 (by the multiplication principle).

§ 2. Basic combinatorial schemes

Arrangements with repetitions

Definition. An arbitrary ordered set of k elements of an n-element set A is called a selection with repetitions of k elements out of n.

1) The elements in the placement can be repeated.

2) Arrangements with repetitions are considered different if they either have a different composition of elements or, in the case of the same composition of elements, differ in the order of placement of the elements.

For example, let $A = \{a, b, c\}$ and we choose $k = 2$ elements.

Let's compose all arrangements of two elements each of the set A: *aa,ab,ac,ba,bb,bc,ca,cb,cc* – we get 9 ($9=3^2$) arrangements with possible repetitions, and arrangements *ab* and *ba* are considered different: they have the same composition of elements, but different order of elements in the arrangement.

Theorem 1.1. The number of arrangements of k elements out of n with possible repetitions is n^k

Proof. Let us assume that from an n-element set A we need to choose ~~a~~ arrangements with possible repetitions of k elements: $(a_1, a_2, \cdots a_k)$. Let's count the number of possible choices. The element $a_1 \in A$ can be chosen in n ways, the element $a_2 \in A$ can be chosen in n ways, the element …, element $a_k \in A$ can be chosen in n ways. Then the number of arrangements with possible repetitions of k elements out of n is equal to:

$|A \times A \times \cdots (k\ times) \cdots \times A| = n \cdot n \cdot \cdots \cdot n = n^k$ (as a consequence of the product rule).

Example 6. How many three-digit numbers can be made:

1) from the digits 2, 3, 7, 8, provided that the digits in the number can be repeated?

2) from the digits 0, 3, 4, provided that the digits in the number can be repeated?

1) Here $A = \{2, 3, 7, 8\}, |A| = 4, k = 3.$ The number of three-digit numbers is equal $4^3 = 64.$

2) Here $A = \{0, 3, 4\}, |A| = 3.$ It should be noted that the first digit of a number cannot be 0, so we can choose the first digit in only two ways, and the other two digits in 3^2 ways ($n = 3$, $k = 2$). Therefore, there are 2·9 =18 desired numbers.

<u>Arrangements without repetitions</u>

Let the set A be finite, $|A| = n$ and the number k is known $(0 \leq k \leq n)$.

Definition. An arbitrary ordered set of k elements of an n-element set A that does not contain repeated elements *is called a non-repeating arrangement of k elements of n*.

1) The elements in the selection cannot be repeated.

2) Non-repeating arrangements are considered different if they either have different composition of elements or, in the case of the same composition of elements, differ in the order of placement of the elements.

The number of arrangements without repetitions is indicated by the symbol A_n^k (read: A from n to k).

Let's find out how many arrangements without repetitions of length k can be made from n elements of the set A.

First, let's analyze the problem using a concrete example. Let a set $A = \{1, 2, 3\}$ be given and let $k = 2.$ Let's make up all arrangements with repetitions, there are 9 of them.

1,1	1,2	1,3
2,1	2,2	2,3
3,1	3,2	3,3

We choose from them the arrangements without repetitions. There are 6 of them.

1,2	1,3.
2,1	2,3
3,1	3,2

Theorem 1.2. The number of arrangements without repetitions from *n* to *k* is equal to $\frac{n!}{(n-k)!}$.

Proof. It is necessary to choose an ordered set of distinct elements $a_1, a_2, \cdots a_k$ from the set A, $|A| = n$.

We can choose the first element a_1 in n ways, the second element a_2 in $(n-1)$ ways, since the first element will already be chosen, and the second element can be chosen out of $(n-1)$ remaining elements, etc. Then we get that

$$A_n^k = n(n-1)(n-2)\cdots(n-k+1).$$

Definition. The factorial of a natural number *n* is the product of all consecutive natural numbers not exceeding the given number n.

The factorial of a number *n* is denoted by $n!$. By definition, $n! = 1 \cdot 2 \cdot \cdots \cdot n$. For example, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. The usual convention is that $0! = 1$ by definition,

The term "factorial" comes from the English word "factor" (multiplier).

The above formula can be transformed as follows:

$$A_n^k = \frac{n!}{(n-k)!}.$$

Example 7. In a volleyball competition, a set of medals is played among 7 participating teams. In how many ways can the medals be distributed?

Here $n = 7, k = 3$. Let's calculate $A_7^3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 5 \cdot 6 \cdot 7 = 210$ ways to distribute medals.

Remark. It is important to remember that when calculating the number of arrangements, the order of ~~the~~ elements is taken into account.

Permutations

Definition. Any arrangement of *n* elements by *n* without repetitions is called a *permutation.*

Permutation is a special case of arrangements without repetitions A_n^k for $k = n$.

Therefore, their number is $P_n = A_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

Example 8.

There are 6 books on a shelf, 4 in mathematics and 2 in physics. In how many ways can these books be arranged on one shelf? In how many ways can the books be arranged on the shelf so that the mathematics books are next to each other? In how many ways can the books be arranged on the shelf so that the physics books are not next to each other?

The number of possible ways to arrange 6 books on a shelf is

$$P_6 = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720.$$

To calculate the number of ways to arrange the books so that the math books are next to each other, let's "glue" the math books together. Then we have 3 "books", the number of ways to arrange them is 3! = 6, and now let's calculate the number of options for placing 4 math books next to each other, when they are next to each other, the number of options is 4! = 24. Then the number of options sought by the product rule is 6·24 = 144.

To calculate the number of ways to arrange the books so that the physics books are not next to each other, first count the number of possible arrangements of the books without restrictions and subtract the number of options when the physics books are next to each other. We have 6! - 5! · 2! = 720 – 240 = 480 options.

Combinations

Definition. *Combinations* of *n* elements by *k* without repetitions are possible arrangements of length *k*, which differ from each other only in composition.

The number of combinations are denoted as C_n^k (read as C from n to k). However, in many ~~of~~ English mathematical publications the notation $\binom{n}{k}$ is used (read as *n*-choose-*k*).

Consider, for example, the set $A = \{1, 2, 3\}$ and choose $k = 2$. Let's write down all ordered sets of elements of the set A, two by two.

1,1	1,2	1,3
2,1	2,2	2,3
3,1	3,2	3,3

The combinations must differ only in the composition of the elements, the order of the elements is not taken into account. Of the written sets of combinations, exactly three combinations are combinations without repetitions: 1,2; 1,3 and 2,3, that is $C_3^2 = 3$.

By definition, we assume that $C_n^0 = C_n^n = 1$ for all integers $n \geq 0$.

Theorem 1.3. The number of combinations from *n* to *k* is $\frac{n!}{k!(n-k)!}$ for $k \geq 1$.

It is clear that the number $C_n^k \leq A_n^k$. For $k < n$, the number of combinations C_n^k can be obtained if, from the number of arrangements without repetitions, we remove arrangements with the same elements, but which differ in the order of elements, this number is equal to $P_k = k!$ Then: $C_n^k = \frac{A_n^k}{P_k} = \frac{n!}{k!(n-k)!}$.

Example 9. In the lottery, a participant draws 6 numbers from a set of numbers from 1 to 49. The order does not matter. The number of possible combinations is:

$$C_{49}^6 = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{6!} = 13983816.$$

Approximately, one chance in 14 million to hit the jackpot!

Remark. In the language of set theory, the number C_n^k *is equal to the number of k-element subsets of the n-element set A when* $0 \leq k \leq n$.

$C_n^0 = 1$ is the number of subsets of an n-element set *A* that do not contain any element, there is only one such subset: it is the empty set.

$C_n^1 = n$ is the number of one-element subsets of n-element set *A*, it is equal to the number of elements of the set *A*.

$C_n^n = 1$ is the number of subsets of an n-element set *A* that contain n elements, there is only one such subset: this is the set *A* itself.

We immediately get the formula:

$$(1+1)^n = C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^{n-1} + C_n^n = 2^n.$$

Example 10. How many 4-element subsets can be chosen from a 7-element set?

$$C_7^4 = \frac{7!}{4!(7-4)!} = \frac{7!}{4!\cdot 3!} = \frac{5\cdot 6\cdot 7}{2\cdot 3} = 35 \text{ subsets.}$$

Example 11. Binary sequences are sequences consisting of zeros and ones. There are 2^n n-digit binary sequences, since each character takes on the values 0 and 1. For example, there are 8 binary sequences of length 3: 000, 001, 010, 011, 100, 101, 110, 111.

1) How many binary sequences of length 12 contain exactly 6 zeros?

2) How many of them have more zeros than ones?

Solution. 1) Six zeros occupy 6 of the 12 places. Therefore, there are $C_{12}^6 = 924$ ways to choose six positions out of 12. This is the desired number.

2) There are $2^{12} - 924$ sequences with an uneven number of zeros and ones. Due to symmetry, exactly half of them, i.e. $\frac{2^{12}-924}{2} = 1586,$ have fewer zeros than ones.

Let's end this section with two simple properties of numbers C_n^k.

Theorem 1.4.

1) $C_n^k = C_n^{n-k}$ for every k, $0 \le k \le n$.

2) $C_{n+1}^k = C_n^k + C_n^{k-1}$ for every k, $0 < k \le n$.

Proof

1) Calculate using the formula:

$$C_n^{n-k} = \frac{n!}{(n-n+k)!(n-k)!} = \frac{n!}{k!(n-k)!} = C_n^k.$$

It is also sufficient to note that choosing *k* out of *n* is the same as choosing $n-k$, which should not be chosen.

2) k items selected from *n+1* items $x_1, x_2, \cdots, x_{n+1}$ may contain or not contain x_{n+1}. If not, then k items are selected from $x_1, x_2, \cdots, x_n$. The number of ways of such selection is C_n^k.

If the k items selected contain x_{n+1}, then the remaining $k-1$ items must be selected from $x_1, x_2, \cdots, x_n$. There are C_n^{k-1} ways to do this. The result now follows from the sum rule (the addition principle).

Another way to prove:

$$\mathrm{C}_n^k + \mathrm{C}_n^{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k)!(n-k+1)} = \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k} + \frac{1}{n-k+1}\right) = \frac{n!(n+1)}{k(k-1)!(n-k)!(n-k+1)} = \frac{(n+1)!}{k!(n-k+1)!} = \mathrm{C}_{n+1}^k.$$

Combinations with repetitions

Now suppose we want to select *k* elements out of *n* with possible repetitions, regardless of order. For example, there are 10 ways to select two elements from the set {1, 2, 3, 4} without considering order with possible repetitions. Here they are: 1,1; 1,2; 1,3; 1,4; 2,2; 2,3; 2,4; 3,3; 3,4; 4,4.

Theorem 1.5. The number of combinations of *k* elements out of *n* with possible repetitions is C_{n+r-1}^r.

Proof

Each selected combination contains x_1 items of the first element, x_2 items of the second element, etc., while $x_1 + x_2 + \cdots + x_n = r$. Therefore, the required number of combinations is equal to the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r$$

in non-negative integers.

We can write the solution to the equation as a binary sequence: $\underbrace{00...0}_{x_1}1\underbrace{00...0}_{x_2}1 00...0 1\underbrace{00...0}_{x_n}$, units are here considered as a transition between elements, provided that they are different.

For example, the solution $x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 1$ of the equation $x_1 + x_2 + x_3 + x_4 = 5$ corresponds to a binary sequence 00110010.

In the binary sequence corresponding to a solution of the equation $x_1 + x_2 + \cdots + x_n = r$, there will be r zeros and $n - 1$ ones. The length of this sequence is $n + r - 1$.

The converse statement is also true: ~~for~~ every binary sequence of length $n + r - 1$, containing *r* zeros and n ones corresponds to a solution of the equation $x_1 + x_2 + \cdots + x_n = r$ in non-negative integers. Thus, also we have proved the following result. In this case, *r* zeros can be located in any $n + r - 1$ places in the sequence and, thus, the number of unordered choices, i.e. the number of combinations C_{n+r-1}^{r} is the desired number.

Theorem 1.6. The number of solutions to the equation $x_1 + x_2 + \cdots + x_n = r$ in non-negative integers is equal to C_{n+r-1}^{r}.

Example 12. The number of solutions to the equation $x + y + z = 17$ in non-negative integers is equal to $C_{17+3-1}^{17} = C_{19}^{17} = C_{19}^{2} = 171$.

Example 13. How many solutions does the equation $x + y + z = 17$ have in positive integers?

Under the conditions of our example $x \geq 1, y \geq 1, z \geq 1$, therefore, we will put

$$x = 1 + u, y = 1 + v, z = 1 + w.$$

Substituting these into the equation, we get $u + v + w = 14$. Now we look for a solution to the new equation in non-negative integers *u*,*v*,*w*. The number of solutions is $C_{14+3-1}^{14} = C_{16}^{2} = 120$.

Example 14. How many binary sequences exist that contain exactly *p* zeros and *q* ones with $q \geq p - 1$ that do not have consecutive zeros?

Let's imagine *q* ones arranged in a row: ⋯1⋯1⋯1⋯.

They form *q+1* cells, in which it is necessary to place zeros (*q-1* cells between the ones and on one each of the edges) *p* zeros must be placed in different cells, so the number of ways to arrange them is C_{q+1}^{p}.

Let's summarize the results of the paragraph in a table:

Number of ways to select r elements from n	*Considering the order*	*Regardless of order*
No repetitions	A_n^r	C_n^r
With possible repetitions	n^r	C_{n+r-1}^r

§ 3. Newton's binomial formula and Pascal's triangle

The word binomial means "two numbers." In mathematics, a binomial is a formula for expanding a power of the sum of two terms into a sum of a non-negative integer power. Newton's binomial is a formula that, for *n* > *0*, has the form:

$$(a+b)^n =$$

$$= \ a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{6}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n =$$

$$= \ a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + C_n^3 a^{n-3}b^3 + \cdots + C_n^{n-1}ab^{n-1} + b^n,$$

where C_n^k is number of combinations of n elements to k:

$$C_n^k = \frac{A_n^k}{P_k} = \frac{n!}{k!\,(n-k)!}.$$

The numbers C_n^k are also called binomial coefficients. Now we will see why. Note that

$$(a+b)^0 = 1,$$

$$(a+b)^1 = a+b,$$

$$(a+b)^2 = a + 2ab + b^2,$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \text{ і т.д.}$$

The coefficients form a triangle called Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

…………………………………………………

or

$$C_0^0 = 1$$

$$C_1^0 = 1 \qquad C_1^1 = 1$$

$$C_2^0 = 1 \qquad C_2^1 = 2 \qquad C_2^2 = 1$$

$$C_3^0 = 1 \qquad C_3^1 = 3 \qquad C_3^2 = 3 \qquad C_3^3 = 1$$

$$C_4^0 = 1 \qquad C_4^1 = 4 \qquad C_4^2 = 6 \qquad C_4^3 = 4 \qquad C_4^4 = 1$$

…………………………………………………………………………………….....

The elements of the r-th row of Pascal's triangle are the binomial coefficients $C_n^r\ (0 \leq r \leq n)$.

The triangle illustrates the following properties of the coefficients:

1) the symmetry of each row of coefficients;

2) each subsequent element of the row is the sum of the two elements above it;

3) the sum of the elements of each row of coefficients is the corresponding power of 2.

Although the binomial formula is named after Isaac Newton (1643-1727), it was known long before him. At the end of the 17th century, Newton himself generalized the binomial formula to the case when the exponent of the power n is any real number. In this case, the binomial is an infinite series. The record of Newton's binomial formula, preserved for posterity, is first found in a book from 1265 by the Persian mathematician Nasir al-Din Tusi, which contains a table of binomial coefficients up to n=12 inclusive.

The binomial formula was known to the Chinese mathematician Yang Hui, who lived in the 13th century, as well as to the Persian mathematician al-Kashi (15th century).

In the middle of the 16th century, the German mathematician Michael Stiefel (1487-1567) described binomial coefficients and also compiled a table of them up to n=18. A detailed study of the properties of the binomial coefficients was carried out by the French mathematician and philosopher Blaise Pascal (1623–1662) in 1654, using them in his work on probability theory. A rigorous proof of the binomial formula for an arbitrary positive integer *n* was given in 1713 by the Swiss mathematician Jacob Bernoulli (1655–1705). Newton's binomial formula for a positive integer n is a special case of the expansion of the function $(1+\mathrm{x})^{\alpha}, \alpha\epsilon\mathbb{R}$ in Taylor's series.

Theorem 1.7 (Binomial Theorem). For every natural value *n*, the following formula is valid:

$$(x+y)^n = C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^n y^n = \sum_{r=0}^{n} C_n^r x^{n-r} y^r \ (*).$$

Proof

Let us present the expression $(x+y)^n$ as a product of n factors:

$(x+y)^n = (x+y)(x+y)\cdots(x+y)$.

The coefficient of $x^{n-r}y^r$ is equal to the number of ways to obtain $x^{n-r}y^r$ by opening the parentheses. Each term obtained by opening the parentheses is the product of one of ,the terms from each couple of parentheses. Thus, $x^{n-r}y^r$ is obtained as many times as there are ways to choose *y* from *r* couples of parentheses (or, which is the same, x from *n* - *r* couples of parentheses). But this is the number of combinations from *n* to *r*. The formula (*) is proved.

Corollary 1. $(1+y)^n = \sum_{r=0}^{n} C_n^r y^r$.

To prove this formula, it suffices to put x = 1 in equation (*).

Corollary 2. Put x = y = 1 into equation (*) and we get

$1 + C_n^1 + C_n^2 + \cdots + C_n^n = 2^n$.

Corollary 3. $C_n^0 - C_n^1 + C_n^2 - \cdots + (-1)^n C_n^n = 0, (n > 0)$.

It suffices to put $\mathrm{x} = 1, \mathrm{y} = -1$ in (*).

Example 15. Compose Pascal's triangle for $n = 7$.

			1					0^{th} line
			1	1				1^{th} line
		1	2	1				
		1	3	3	1			
	1	4	6	4	1			
	1	5	10	10	5	1		
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	7^{th} line

Theorem 1.8. For everyone $m \geq 0$ and $n \geq 0$

$$C_m^m + C_{m+1}^m + \cdots + C_{m+n}^m = C_{m+n+1}^{m+1}.$$

Proof

Let's apply the property of combinations: $C_{m+n+1}^{m+1} = C_{m+n}^m + C_{m+n}^{m+1}$, знову застосуємо цю властивість: $C_{m+n+1}^{m+1} = C_{m+n}^m + C_{m+n-1}^m + C_{m+n-1}^{m+1}$ and so on. Finally $C_{m+n+1}^{m+1} = C_{m+n}^m + C_{m+n-1}^m + \cdots + C_{m+1}^m + C_{m+1}^{m+1}$.

Since $C_{m+1}^{m+1} = 1 = C_m^m$, we get the desired result.

The binomial theorem allows us to obtain some identities.

Example 16. Consider the identity

$$(1 + x)^n (1 + x)^n = (1 + x)^{2n},$$

that is

$$(C_n^0 + C_n^1 x + \cdots + C_n^n x^n)(C_n^0 + C_n^1 + \cdots + C_n^n x^n) = \sum_{r=0}^{2n} C_{2n}^r x^r.$$

Equating the coefficients at x^n in both parts of this identity, we obtain the equality

$$C_n^0 C_n^n + C_n^1 C_n^{n-1} + \cdots + C_n^n C_n^0 = C_{2n}^n,$$

which can be written as

$$(C_n^0)^2 + (C_n^1)^2 + \cdots + (C_n^n)^2 = C_{2n}^n.$$

For example, for $n = 4$ we get

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = C_8^4.$$

Similarly, we obtain the result for the alternating sum.

Example 17. Use identity $(1 - x^2)^n = (1 - x)^n(1 + x)^n$ to find the sum $(C_n^0)^2 - (C_n^1)^2 + (C_n^2)^2 - \cdots + (-1)^n C_n^n$.

Consider the coefficient before x^n in both sides of the equality. The right-hand side has the form

$$(1 - x)^n(1 + x)^n = (1 - C_n^1 x + C_n^2 x^2 - \cdots + (-1)^n C_n^n x^n)(1 + C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n).$$

The coefficient at x^n is

$$\sum_{r+s-n} (-1)^r C_n^r C_n^s = \sum_{r=0}^{n} (-1)^r C_n^r C_n^{n-r} = \sum_{r=0}^{n} (-1)^r (C_n^r)^2.$$

The coefficient at x^n on the left-hand side is zero if n is odd, and is equal to $(-1)^{\frac{n}{2}} C_n^{\frac{n}{2}}$, if n is even. So we have

$$(C_n^0)^2 - (C_n^1)^2 + (C_n^2)^2 - \cdots + (-1)^n (C_n^n)^2 = \begin{cases} (-1)^{\frac{n}{2}} C_n^{\frac{n}{2}}, & \text{if n is even.} \\ 0, & \end{cases}$$

For example,

$$(C_5^0)^2 - (C_5^1)^2 + (C_5^2)^2 - (C_5^3)^2 + (C_5^4)^2 - (C_5^5)^2 = 0,$$

$$(C_4^0)^2 - (C_4^1)^2 + (C_4^2)^2 - (C_4^3)^2 + (C_4^4)^2 = 1 - 16 + 36 - 16 + 1 = 6 = C_4^2.$$

$$(C_6^0)^2 - (C_6^1)^2 + (C_6^2)^2 - (C_6^3)^2 + (C_6^4)^2 - (C_6^5)^2 + (C_6^6)^2 = -20 = -C_6^3.$$

§ 4. Partitions of a set.

Definition. A partition of a set *S* is a nonempty set of subsets $S_1, \dots, S_r$ of the set *S* that are pairwise disjoint and whose union is equal to *S*. The subsets S_i are called *parts of the partition*.

For example, $\{1,2,4\} \cup \{3,6\} \cup \{5\}$ is a partition of the set {1, 2, 3, 4, 5, 6} into three parts. Note that the order in which the parts are arranged and the order of the elements in the part do not play a role.

Example 18.

In a game of bridge, 52 cards are dealt to four players, who receive 13 cards each. In how many ways can this be done?

The number of ways to choose 13 cards out of 52 is C_{52}^{13}. Out of the remaining 39 cards, there are C_{39}^{13} ways to choose the next 13 cards. Then there are C_{26}^{13} ways to choose 13 out of the remaining 26 cards. After that, there are 13 cards left. Thus, we have

$$C_{52}^{13} \cdot C_{39}^{13} \cdot C_{26}^{13} = \frac{52!}{13!\,39!} \cdot \frac{39!}{13!\,26!} \cdot \frac{26!}{13!\,13!} = \frac{52!}{(13!)^4}$$

ways to split the deck. But not all of these ways are different. Each split occurs 4! times. Finally, the number of possible splits is $\frac{52!}{(13!)^4 4!}$.

There is another method for solving this problem.

Consider a string of 52 elements, which are grouped into groups of 13.

$$(\cdots)\,(\cdots)\,(\cdots)\,(\cdots).$$

The cards can be arranged in 52! ways. There are 13! ways to arrange 13 cards in each group. So we have to divide 4 times by 13!, that is, by $(13!)^4$. The number of ways to arrange the groups is 4!. So we have to divide by 4! again. We get the same answer $\frac{52!}{(13!)^4 4!}$.

Theorem 1.9. A set of $m \cdot n$ elements can be partitioned into *m n*–element subsets in $\frac{(mn)!}{(n!)^m m!}$ ways.

Corollary. *2m* elements can be divided into *m* pairs in $\frac{(2m)!}{(2!)^m m!}$ ways.

Example 19. How many ways are there to form pairs of 16 participating teams in a soccer cup?

$$\frac{16!}{2^8 8!} = 2027025.$$

Similar considerations can be used when the parts of the partition contain different numbers of elements.

Example 20. How many ways are there to divide 25 students into 4 groups of 3, 2 groups of 4, and 1 group of 5?

There are 25! ways to arrange 25 students. But we consider the same ways with the same composition of the groups, even if the order is different. Therefore, the total number of ways must be divided by $(3!)^4 \cdot (4!)^2 \cdot 5!$. But if the groups are swapped, we get the same division, so this number must be divided by 4! · 2!. That is, the number of ways to arrange is

$$\frac{25!}{(3!)^4 \cdot (4!)^2 \cdot 5! \cdot 4! \cdot 2!}.$$

Definition.. A partition of an *n*-element set consisting of α_i subsets of i elements, where $1 \le i \le n$, $\sum_{i=1}^{n} i\alpha_i = n$ is called a partition of the type $1^{\alpha_1} \cdot 2^{\alpha_2} \cdot \dots \cdot n^{\alpha_n}$.

Generalizing the result of the example, we obtain the following theorem.

Theorem 1.10. The number of partitions of type $1^{\alpha_1} \cdot 2^{\alpha_2} \cdot \dots \cdot n^{\alpha_n}$ is equal to

$$\frac{n!}{\prod_{i=1}^{n} (i!)^{\alpha_i} \alpha_i!}.$$

§ 1.5. Stirling numbers of the second kind. Bell numbers.

We will talk about dividing a set into a given number of parts.

Definition. Let *S (n, k)* denote the number of ways to partition a set of *n* elements into *k* parts. The numbers *S (n, k)* are called *Stirling numbers of the second kind.*

It is obvious that for all *n* the following holds: $S(n,1) = S(n,n) = 1$.

Example 21. Let's find $S(4,2)$:

$\{1\} \cup \{2,3,4\}$; $\{2\} \cup \{1,3,4\}$; $\{3\} \cup \{1,2,4\}$; $\{4\} \cup \{1,2,3\}$;

$\{1,2\} \cup \{3,4\}$; $\{2,4\} \cup \{1,3\}$; $\{1,4\} \cup \{2,3\}$.

So $S(4,2) = 7$.

To find $S(n,k)$ for large *n* and *k*, the following theorem is used:

Theorem 1.11. $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ для $1 < k < n$.

Proof

In any partition of the set {1, 2,…, *n*} into *k* parts, element n may appear as a separate subset or as a part of a subset containing more than one element. If it is itself a subset, then the other *n - 1* elements must form a partition of the set {1, 2,…, *n - 1*} into *k - 1* subsets. This can be done in $S(n-1,k-1)$ ways.

On the other hand, if element *n* is in a subset of size ≥ 2, then we can say that we have partitioned the set {1, 2,…, *n - 1*} into *k* subsets (which can be done in $S(n-1,k)$ ways), and then added *n* to one of the sets of the resulting partition, which can be done in *k* ways.

Thus, $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

Example 22. Let us calculate again $S(4,2) = S(3,1) + 2S(3,2) = 1 + 2\big(S(2,1) + 2S(2,2)\big) = 1 + 2(1+2) = 7$.

Theorem 1.12. $For\ all\ n \geq 2,\ we\ have\ S(n,2) = 2^{n-1} - 1.$

Proof

We prove by induction on n.

For $n = 2$ $S(2,2) = 2^{n-1} - 1 = 1$.

Let $S(k,2) = 2^{k-1} - 1$ for all $k \geq 2$.

Then $S(k+1,2) = S(k,1) + 2S(k,2) = 1 + 2(2^{k-1} - 1) = 2^k - 1 = 2^{(k+1)-1} - 1$.

So, the statement is true for all $n \geq 2$.

Definition. Bell numbers are numbers calculated by the formula

$$B(n) = \sum_{k=1}^{n} S(n,k).$$

The number $B(n)$ is the total number of ways to partition an *n*-element set. By definition, we have: $B(0) = S(0,0) = 1$.

Theorem 1.13. *For* $n \geq 1$

$$B(n) = \sum_{k=1}^{n-1} C_{n-1}^{k} B(k).$$

Proof

The *n*th element of a set can be partitioned into one of the subsets along with ∃ j $(j \geq 0)$ other elements. Then the remaining *n - 1- j* elements can be partitioned in $B(n-1-j)$ ways. Thus

$$B(n) = \sum_{j=0}^{n-1} C_{n-1}^{j} B(n-1-j).$$

Making the substitution $n - 1 - j = k$, we get

$$B(n) = \sum_{k=0}^{n-1} C_{n-1}^{n-1-k} B(k)$$

or

$$B(n) = \sum_{k=0}^{n-1} C_{n-1}^{k} B(k).$$

Example 23. $B(9) = \sum_{k=0}^{9} C_9^k B(k) = 21147.$

Table 1 lists the first few Stirling numbers, and the closely related Bell numbers in the last column.

Table 1

*n**k*	1	2	3	4	5	6	7	8	*B*(*n*)
1	1								1
2	1	1							2
3	1	3	1						5
4	1	7	6	1					15

5	1	15	25	10	1				52
6	1	31	90	65	15	1			203
7	1	63	301	350	140	21	1		877
8	1	127	966	1701	1050	266	28	1	4140

§ 1.6. Counting functions

Consider the mappings (functions) that act from a set X, $|X| = m$, to a set Y, $|Y| = n$, $f : X \to Y$. Recall that the notation $|X| = m$ means that the set X contains m elements. In other words, we can say that X is an m-element set and Y is an n-element set.

Let us determine how many functions there are that act from X to Y.

Theorem 1.14. If $|X| = m$, $|Y| = n$, then the number of functions $f: X \to Y$ is equal to n^m.

Proof

Let X={$x_1, x_2, \dots,$ x_m}. If $f: X \to Y$, then each function f can be associated with an ordered set ($y_1, y_2, \dots,$ y_n), where $y_i = f(x_i), i = \overline{1, m}$. Different sets will correspond to different functions, and *vice versa*.

The 1st element of the set can take n values (Y is an n-element set).

The 2nd element of the set can take n values, etc.

According to the product rule, the number of different sets will be $n \cdot n \dots \cdot n$ (m times) $= n^m$. The theorem is proved.

Sometimes it is convenient to formulate this theorem in other terms

Theorem 1.15. The number of possible arrangements of m different objects in n different boxes is n^m.

Theorem 1.16. The number of possible words of length m in an alphabet of n characters is n^m.

. Let's explain with an example.

Example 24. Let's calculate how many functions there are that act from $X=\{0,1\}$ to $Y=\{2,3,4\}$. Here $m=2$, $n=3$, the number of functions is $n^m = 3^2=9$. Let's describe the functions explicitly.

f_1	X	0	1
	Y	2	2
f_2	X	0	1
	Y	2	3
f_3	X	0	1
	Y	3	2
f_4	X	0	1
	Y	3	3
f_5	X	0	1
	Y	3	4

f_6	X	0	1
	Y	4	3
f_7	X	0	1
	Y	4	4
f_8	X	0	1
	Y	2	4
f_9	X	0	1
	Y	4	2

Let's impose certain conditions on the functions $f: X \to Y$.

Definition. .

A function $f: X \to Y$ is called injective if the condition $x_i \neq x_j$ implies $f(x_i) \neq f(x_j)$, i.e. different elements have different images.

In terms of placing objects in boxes, the injectivity of a function will mean that each box contains at most one object.

In terms of words in a given alphabet, each symbol (letter) in a word occurs at most once.

Recall that the number of arrangements without repetitions is

$$A_n^k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Theorem 1.17. If $|X| = m,\ |Y| = n,$ then the number of injective functions $f: X \to Y$ equals A_n^m.

Proof

Let X={$x_1, x_2, \ldots, x_m$}, if $f: X \to Y$, then each function f can be associated with an ordered set $(y_1, y_2, \ldots, y_m)$, where $y_i = f(x_i), i = \overline{1, m}$. Since all functions are injective, all elements in the set are distinct.

The 1st element of the set can take n values (*Y* is an *n*-element set).

The 2nd element of the set can take *n - 1* values.

The 3rd element of the set can take *n - 2* values, etc.

According to the product rule, the number of different sets will be equal to $n(n-1)(n-2) \ldots (n-m+1) = A_n^m$. The theorem is proved.

In terms of placement: m objects can be placed in *n* different boxes A_n^m ways, provided that each box contains at most one object.

In terms of words in a given alphabet: the number of words of length m, in which all characters (letters) are different, in an alphabet of *n* characters is A_n^m.

Example 25. The number of injective functions from example 24 is equal to $A_3^2 = 3(3-1) = 6$. These are functions $f_2, f_3, f_5, f_6, f_8, f_9$.

Consider permutations of a finite set as injective mappings of the set onto itself.

Definition. A permutation of a finite set is a one-to-one mapping of this set onto itself.

Since the elements of a finite set can be enumerated, we can define a permutation as follows:

Definition. A permutation of the *n*th degree is a one-to-one mapping of the set {1, 2, …, n} onto itself.

A permutation is usually written as

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \varphi(1) & \varphi(2) & \varphi(3) & \ldots & \varphi(n) \end{pmatrix}, \quad \text{where } \varphi(i) \in \{1, 2, \ldots, n\},\ i = 1, 2, \ldots, n.$$

In this case $X = Y$ and $|X| = |Y| = n$. According to the theorem, the number of permutations is $A_n^n = n!$

Recall that the *image of a function* $f : X \to Y$ is the set of elements from *Y* for which there exists $x \in X$:

$$Im\, f = E(f) = \{y \in Y : \exists x \in X\ f(x) = y\}.$$

For each function $f : X \to Y$ the image is a subset of the set Y ($Im\, f \subseteq Y$). Let us ask the following question: how many functions have a domain that is a k - element subset of the set Y $(0 \leq k \leq n)$?

If there are *k* functions $f: X \to Y$, then the set *X* can be partitioned into k subsets such that the i-th subset consists of those elements of the set *X* that are mapped by the function *f* into the i-th element of *Im f*. Thus, the function, $f : X \to Y$ whose domain size is *k*, can be constructed as follows:

1) we divide *X* into *k* parts $X_1, X_2, \dots, X_k$ (this can be done in *S (m ,k)* number of ways);
2) we choose a domain of values in *Y* of size *k* (this can be done in C_n^k ways);
3) we associate one element from the image to each X_i (this can be done in *k!* ways).

Thus, the number of functions $f : X \to Y$ with a domain of values of size *k* is equal to $S\ (m, k) C_n^k\, k!$.

Example 26. Let's illustrate the above with an example.

Let $X = \{0, 1, 2\}$, $Y = \{0, 1, 2, 3\}$, $m = 3, n = 4$. Let's choose $k = 2$.

Let's divide the set *X* into 2 parts:

$$\{0\} \cup \{1, 2\}\ or\ \{0, 1\} \cup \{2\}\ or\ \{1\} \cup \{0, 2\}.$$

Let's divide the set Y into 2 parts:

$$\{0, 1\}\ or\ \{0, 2\}\ or\ \{0, 3\}\ or\ \{1, 2\}\ or\ \{1, 3\}\ or\ \{2, 3\}.$$

Then the number of functions with a range of values of size 2 is

$$S\ (3, 2) \cdot C_4^2 \cdot 2 = 3 \cdot 6 \cdot 2 = 36.$$

Theorem 1.18. *Ler* $|X| = m$ *and* $|Y| = n$*, where* $m \geq 1, n \geq 1$. *Then*

1) The number of functions $f : X \to Y$ *with the range of size k is*$S\ (m, k) C_n^k\, k!$.

2) The equality

$n^m = \sum_{k=1}^{n} S\,(m,k) C_n^k\, k!$ (1)

holds.

Proof

1) is proved above;

Note that the number of surjections, i.e. functions $f : X \to Y$, whose range is $Im\, f = Y$, is equal to $n!\, S\,(m,n)$ (we put *n* = *k*).

2) *k* takes values from 1 to *n*, and, as shown earlier, the number of functions is n^m.

Example 27. Let us check whether equality (1) holds for the case when $m = 5, n = 4$.

$$\sum_{k=1}^{n} S\,(5,k) C_k^k\, k! = 4S\,(5,1) + 12S\,(5,2) + 24S\,(5,3) + 24S\,(5,4) =$$

$$= 4 + 180 + 600 + 240 = 1024 = 4^5.$$

Note that, assuming by definition *S (m, 0) = 0* for all $m \geq 1$ and *S (0, 0) = 1*, we can rewrite (1) as follows:

$$n^m = \sum_{k=0}^{n} S\,(m,k) C_n^k\, k!$$

The following lemma and theorem provide a quick way to obtain an expression for Stirling numbers of the second kind in terms of binomial coefficients

Lemma. If $j \leq k \leq i$, то $C_i^k C_k^j = C_i^j C_{i-j}^{k-j}$.

Proof

$$C_i^k C_k^j = \frac{i!}{(i-k)!\,k!} \cdot \frac{k!}{(k-j)!\,j!} = \frac{i!}{(i-k)!\,(k-j)!\,j!}$$

$$= \frac{i!}{j!\,(i-j)!} \cdot \frac{(i-j)!}{(i-k)!\,(k-j)!} = C_i^j C_{i-j}^{k-j}.$$

Theorem 1.19. L*et* A *be a matrix of size* $(n+1)\times(n+1)$ *with elements* $a_{ij}C_i^j$ $(i,j=0,\dots,n)$, *and* B *be a matrix of the same size with elements* $b_{ij} = (-1)^{j+i}C_i^j$. *Then* $BA = I$.

Proof

$$A = \begin{pmatrix} C_0^0 & C_0^1 & \dots & C_0^n \\ C_1^0 & C_1^1 & \dots & C_1^n \\ \dots & \dots & \dots & \dots \\ C_n^0 & C_n^1 & \dots & C_n^n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n & \frac{(n+1)}{2}n & \dots & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} C_0^0 & -C_0^1 & C_0^2 & \dots \\ -C_1^0 & C_1^1 & -C_1^2 & \dots \\ C_2^0 & -C_2^1 & C_2^3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & -n & \frac{(n+1)}{2}n & \dots & 1 \end{pmatrix}.$$

For the *(i,i)*-th element of the matrix *BA* we have (assuming that $C_n^r = 0$, if $r > m$):

$$(BA)_{ii} = \sum_k (-1)^{i+k}\, C_i^k C_k^i = (-1)^{i+i}(C_i^i)^2 = 1.$$

If $i \neq j$, then

$$(BA)_{ij} = \sum_k (-1)^{i+k}\, C_i^k C_k^i = (-1)^i \sum_k (-1)^k\, C_i^k C_k^i = (-1)^i \sum_k (-1)^k\, C_i^j C_{i-j}^{k-j}$$

$$= (-1)^i C_i^j \sum_k (-1)^k\, C_{i-j}^{k-j}.$$

For $k < j$ $C_{i-j}^{k-j} = 0$, so

$$(BA)_{ij} = \sum_{l=0}^{i-j} C_{i-j}^l (-1)^{j+l} = (-1)^j \sum_{l=0}^{i-j} C_{i-j}^l (-1)^l = (-1)^j (1-1)^{i-j} = 0.$$

Corollary 1. *If* $a_n = \sum_{k=0}^n C_n^k b_k$ *for all n ≥ 0, then*

$$b_n = \sum_{k=0}^{n} (-1)^{n+k} C_n^k a_k.$$

The following corollary is a formula for calculating Stirling numbers of the 2nd kind via binomial coefficients.

Corollary 2. *For all* $m \geq 1, n \geq 0, m \geq n$:

$$S\,(m, n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} C_n^k k^m.$$

To prove the formula, it is sufficient to put in

$a_k = k^m b_k = \; k!\; S\,(m, k)$

in Corollary 1.

Example 28.

$$S\,(5, 3) = \frac{1}{3!} \left(\sum_{k=0}^{3} (-1)^3 - k C_3^k k^5 \right) = \frac{1}{6} (-0 + 3 - 3 \cdot 2^5 + 3^5) = 2^5.$$

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