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THREE SPECTRA PROBLEMS FOR STAR GRAPH OF STIELTJES STRINGS

A. DUDKO AND V. PIVOVARCHIK

This paper is dedicated to Yu. M. Arlinskii's 70th anniversary

Abstract. The (main) spectral problem for a star graph with three edges composed of Stieltjes strings is considered with the Dirichlet conditions at the pendant vertices. In addition we consider the Dirichlet-Neumann problem on the first edge (Problem 2) and the Dirichlet-Dirichlet problem on the union of the second and the third strings (Problem 3). It is shown that the spectrum of the main problem interlace (in a non-strict sense) with the union of spectra of Problems 2 and 3. The inverse problem lies in recovering the masses of the beads (point masses) and the lengths of the intervals between them using the spectra of the main problem and of Problems 2 and 3. Conditions on three sequences of numbers are proposed sufficient to be the spectra of the main problem and of Problems 2 and 3, respectively.

1. Introduction

Boundary value problems on graphs consisting of Stieltjes strings (elastic massless threads bearing point masses, in other words, beads) are natural generalizations of boundary value problems on a single interval and are often used in the theory of vibrations of nets. Direct and inverse problems for a single Stieltjes string were solved in [8].

Finite dimensional inverse spectral problems for star graphs were solved in [2], [19], [20]. By inverse spectral problem we mean recovering the masses of the beads and the lengths of the intervals between them using certain spectral data.

It is clear that the spectrum of the spectral problem on a star graph does not determine uniquely the masses of the beads and the lengths of the intervals between them. The authors of papers [2], [19], [20] used the spectrum of a problem on the whole star graph together with the spectra of boundary value problems on the edges as the given data.

Nowadays, finite dimensional spectral problems complete actual topics of investigation for physicists and mathematicians (see, e.g. [6], [7], [9], [15], [18], [19]) as well as their infinite dimensional analogues (in quantum graph theory), see, e.g. [1], [12], [14], [17].

Finite dimensional spectral problems appear in the theory of vibrations of mechanical systems which have forms of graphs (see, [6], [7], [9], [15], [18], [19]) and in the theory of synthesis of electrical circuits (see, [5], [10]).

In the present paper we consider a star graph with three edges. The spectral problem on the whole graph is the one with the Dirichlet conditions at the graphs pendant vertices and the continuity and balance of forces conditions at the central vertex. Also we consider Dirichlet-Neumann problem (the Dirichlet condition at the left end and the Neumann condition at the right end) on the first edge and the third problem which is Dirichlet-Dirichlet problem (the Dirichlet condition at both ends) on the union of the second and the third strings.

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Our inverse problem lies in recovering of masses of the beads and the lengths of the intervals between them, using the spectra of the three spectral problems described in the above paragraph and the total lengths of the strings. Solving of this inverse problem lies in:
1) establishing conditions on three sequences of real numbers necessary and sufficient to be the spectra of three problems described above;
2) proving uniqueness of the solution (if it is unique);
3) finding a method for recovering the masses and the lengths of the intervals.

We solve the inverse problem in Section 3, while in Section 2 we solve the direct problem, i.e. we describe mutual location of the spectra of three problems described above. This we need to compare the conditions on three sequences of real numbers described in Section 3 as sufficient conditions with the necessary conditions described in Section 2. We show that the three spectra and the total lengths of the strings uniquely determine the masses of the beads and the lengths of the intervals between them on the first string. An algorithm of recovering the masses and the lengths of the intervals between them is also given in Section 3.

2. Direct spectral problem

We consider three Stieltjes strings which are joined in one point (see Fig. 1) to compose a star graph. The joining point of the obtained star graph is free of beads. The pendant vertices of the obtained star graph are fixed. We measure distances from the pendant vertices on each edge. Starting indexing from the pendant vertices, n_j beads of masses $m_k^{(j)} > 0$, $k = 1, 2, \ldots, n_j$, are positioned on the j-th string, $j = 1, 2, 3$, which divide the jth string into $n_j + 1 \quad (n_j \geq 1)$ subintervals the lengths of which are denoted by $l_k^{(j)} > 0 \quad (k = 0, 1, \ldots, n_j)$ again starting indexing from the free ends. In particular, $l_0^{(j)}$ is the distance on the j-th string between the fixed endpoint and $m_1^{(j)}$, $l_k^{(j)}$ for $(k = 1, 2, \ldots, n_j - 1)$ is the distance between the beads of masses $m_k^{(j)}$ and $m_{k+1}^{(j)}$, and $l_{n_j}^{(j)}$ is the distance on the j-th string between the bead of mass $m_{n_j}^{(j)}$ and the point of joining of the strings. The tension of each thread is assumed to be equal to 1.

The transverse displacement of the bead of mass $m_k^{(j)}$ at the time t is denoted by $v_k^{(j)}(t) \quad (k = 1, 2, \ldots, n_j, \quad j = 1, 2, 3)$. For convenience, we denote by $v_0^{(j)}$ the transversal displacement of the pendant endpoints and by $v_{n_j+1}^{(j)}(t)$ the transversal displacement at the point of joining considered as a point on j-th string.

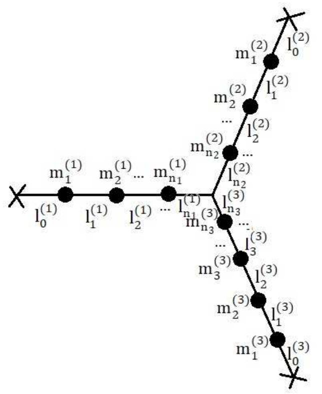

Figure 1.

The total length of jth string is denoted by l_j:

$$l_j = \sum_{k=0}^{n_j} l_k^{(j)}.$$

Newton's law gives the following equations of motion for the beads:

$$\frac{v_k^{(j)}(t) - v_{k+1}^{(j)}(t)}{l_k^{(j)}} + \frac{v_k^{(j)}(t) - v_{k-1}^{(j)}(t)}{l_{k-1}^{(j)}} + m_k^{(j)} v_k^{(j)\prime\prime}(t) = 0$$

$$(k = 1, 2, \ldots, n_j, \quad j = 1, 2, 3).$$

From the continuity of the strings at the central vertex of the graph we obtain

$$v_{n_1+1}^{(1)}(t) = v_{n_2+1}^{(2)}(t) = v_{n_3+1}^{(3)}(t)$$

and the balance of forces at this point leads to

$$\sum_{j=1}^{3} \frac{v_{n_j}^{(j)}(t) - v_{n_j+1}^{(j)}(t)}{l_{n_j}^{(j)}} = 0.$$

We impose the Dirichlet condition at the beginning of each string

$$v_0^{(1)}(t) = v_0^{(2)}(t) = v_0^{(3)}(t) = 0,$$

which means fixing the pendant vertices of the graph.

Substituting $v_k^{(j)}(t) = u_k^{(j)} e^{i\lambda t}$ and changing the spectral parameter for $z = \lambda^2$ we obtain the following recurrences for the amplitudes $u_k^{(j)}$ of vibrations:

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)} z u_k^{(j)} = 0 \quad (k = 1, 2, \ldots, n_j, \quad j = 1, 2, 3), \tag{1}$$

$$u_{n_1+1}^{(1)} = u_{n_2+1}^{(2)} = u_{n_3+1}^{(3)}, \tag{2}$$

$$\sum_{j=1}^{3} \frac{u_{n_j}^{(j)} - u_{n_j+1}^{(j)}}{l_{n_j}^{(j)}} = 0, \tag{3}$$

$$u_0^{(1)} = u_0^{(2)} = u_0^{(3)} = 0. \tag{4}$$

Denote by $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$ the spectrum of problem (1)–(4).

Also we consider the spectral problem for the first string with Dirichlet condition at the left end and with Neumann condition (which means that the end is free to move in the direction orthogonal to the equilibrium position of the string) at the right end (see Fig. 2):

$$\frac{u_k^{(1)} - u_{k+1}^{(1)}}{l_k^{(1)}} + \frac{u_k^{(1)} - u_{k-1}^{(1)}}{l_{k-1}^{(1)}} - m_k^{(1)} z u_k^{(1)} = 0 \quad (k = 1, 2, \ldots, n_1), \tag{5}$$

$$u_0^{(1)} = 0, \tag{6}$$

$$u_{n_1+1}^{(1)} = u_{n_1}^{(1)}. \tag{7}$$

Denote by $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ the spectrum of problem (5)–(7).

In addition to problems (1)–(4) and (5)–(7), we consider the problem that occurs when one of the ends of the second string is connected to one of the ends of the third string, and at the pendant vertices we impose Dirichlet conditions (see Fig. 3):

$$\frac{u_k^{(j)}-u_{k+1}^{(j)}}{l_k^{(j)}}+\frac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}-m_k^{(j)}zu_k^{(j)}=0 \quad (k=1,2,\dots,n_j, j=2,3), \tag{8}$$

$$u_0^{(2)}=u_0^{(3)}=0, \tag{9}$$

$$u_{n_2+1}^{(2)}=u_{n_3+1}^{(3)}. \tag{10}$$

$$\sum_{j=2}^{3}\frac{u_{n_j}^{(j)}-u_{n_j+1}^{(j)}}{l_{n_j}^{(j)}}=0. \tag{11}$$

Denote by $\{\mu_k\}_{k=1}^{n_2+n_3}$ the spectrum of problem (8)–(11).

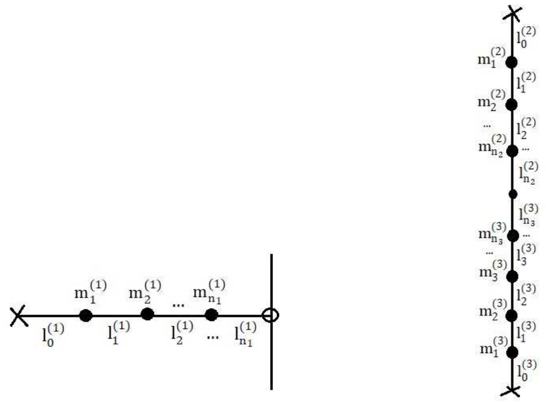

FIGURE 2

Following [8] we look for a solution to problems (1)–(4), (5)–(7), (8)–(11) in the form:

$$u_k^{(j)}=R_{2k-2}^{(j)}(z)u_1^{(j)} \quad (k=1,2,\dots,n_j, \quad j=1,2,3), \tag{12}$$

where $R_{2k-2}^{(j)}(z)$ are polynomials of degree $k-1$.

According to [8] the polynomials $R_k^{(j)}(z)$ satisfy the recurrences

$$R_{2k-1}^{(j)}(z)=-zm_k^{(j)}R_{2k-2}^{(j)}(z)+R_{2k-3}^{(j)}(z),$$
$$R_{2k}^{(j)}(z)=l_k^{(j)}R_{2k-1}^{(j)}(z)+R_{2k-2}^{(j)}(z)$$
$$(k=1,2,\dots,n_j, \quad j=1,2,3)$$

and the initial conditions

$$R_{-1}^{(j)}(z)=\frac{1}{l_0^{(j)}}, \quad R_0^{(j)}(z)=1.$$

Substituting (12) into (2), (3), we obtain

$$R_{2n_1}^{(1)}(z)u_1^{(1)}=R_{2n_2}^{(2)}(z)u_1^{(2)}=R_{2n_3}^{(3)}(z)u_1^{(3)},$$

$$R_{2n_1-1}^{(1)}(z)u_1^{(1)}+R_{2n_2-1}^{(2)}(z)u_1^{(2)}+R_{2n_3-1}^{(3)}(z)u_1^{(3)}=0,$$

or in matrix form

$$\begin{pmatrix} R^{(1)}_{2n_1}(z) & -R^{(2)}_{2n_2}(z) & 0 \\ 0 & R^{(2)}_{2n_2}(z) & -R^{(3)}_{2n_3}(z) \\ R^{(1)}_{2n_1-1}(z) & R^{(2)}_{2n_2-1}(z) & R^{(3)}_{2n_3-1}(z) \end{pmatrix} \begin{pmatrix} u^{(1)}_1 \\ u^{(2)}_1 \\ u^{(3)}_1 \end{pmatrix} = 0. \tag{13}$$

The sequence of eigenvalues of problem (1)–(4) coincides with the sequence of values of the spectral parameter z, for which system (13) has a nontrivial solution, that is, for which the determinant of the matrix of the system is 0

$$\begin{vmatrix} R^{(1)}_{2n_1}(z) & -R^{(2)}_{2n_2}(z) & 0 \\ 0 & R^{(2)}_{2n_2}(z) & -R^{(3)}_{2n_3}(z) \\ R^{(1)}_{2n_1-1}(z) & R^{(2)}_{2n_2-1}(z) & R^{(3)}_{2n_3-1}(z) \end{vmatrix} = 0.$$

Thus, the spectrum $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$ of problem (1)–(4) coincides with the sequence of solutions of the equation

$$\begin{aligned} \Phi(z) :&= R^{(1)}_{2n_1}(z)R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(1)}_{2n_1-1}(z)R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3}(z) \\ &+ R^{(1)}_{2n_1}(z)R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z) = 0. \end{aligned} \tag{14}$$

We call $\Phi(z)$ the characteristic polynomial of problem (1)–(4).

The spectrum $\{\nu^{(1)}_k\}_{k=1}^{n_1}$ of problem (5)–(7) coincides with the sequence of solutions of the equation

$$R^{(1)}_{2n_1-1}(z) = 0. \tag{15}$$

The left side of equation (15) is the characteristic polynomial of problem (5)–(7).

Substituting (12) into (8)–(11), we obtain

$$R^{(2)}_{2n_2}(z)u^{(2)}_1 = R^{(3)}_{2n_3}(z)u^{(3)}_1,$$

$$R^{(2)}_{2n_2-1}(z)u^{(2)}_1 + R^{(3)}_{2n_3-1}(z)u^{(3)}_1 = 0,$$

or in the matrix form

$$\begin{pmatrix} R^{(2)}_{2n_2}(z) & -R^{(3)}_{2n_3}(z) \\ R^{(2)}_{2n_2-1}(z) & R^{(3)}_{2n_3-1}(z) \end{pmatrix} \begin{pmatrix} u^{(2)}_1 \\ u^{(3)}_1 \end{pmatrix} = 0.$$

The spectrum $\{\mu_k\}_{k=1}^{n_2+n_3}$ of problem (8)–(11) coincides with the sequence of solutions of the equation

$$\Psi(z) := R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z) = 0.$$

Here $\Psi(z)$ is the characteristic polynomial of problem (8)–(11). We denote by $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$ the union of the spectra of problems (5)–(7) and (8)–(11), i.e. $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}=\{\nu^{(1)}_k\}_{k=1}^{n_1} \bigcup \{\mu_k\}_{k=1}^{n_2+n_3}$.

We index elements of $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$ in the nondecreasing order such that

$$\xi_1 \le \xi_2 \le \cdots \le \xi_{n_1+n_2+n_3}.$$

Definition 2.1. (see e.g. [16], Definition 5.1.20). A function $\omega(z)$ is said to be Nevanlinna (or R - function in terms of [11]), if:
1) the function $\omega(z)$ is analytic in the half-planes $\mathrm{Im} z > 0$ and $\mathrm{Im} z < 0$;
2) $\omega(\overline{z}) = \overline{\omega(z)}$;
3) $\mathrm{Im} z \ \mathrm{Im}\omega(z) \ge 0$ for $\mathrm{Im} z \ne 0$.

Definition 2.2.(see [11] or [16], Definition 5.1.24). A Nevanlinna function $\omega(z)$ is said to be an S-function, if $\omega(z) > 0$ for $z < 0$.

Definition 2.3. An S-function is called S_0-function, if $|\omega(0)| < \infty$.

Lemma 2.4. *Let $f_1(z)$ and $f_2(z)$ be S_0-functions. Then $f_1(z)+f_2(z)$ and $\left(f_1^{-1}(z)+f_2^{-1}(z)\right)^{-1}$ are also S_0-functions.*

Proof. Since $\mathrm{Im}z\ \mathrm{Im}\ (f_1(z)+f_2(z)) = \mathrm{Im}z\ \mathrm{Im}f_1(z)+\mathrm{Im}z\ \mathrm{Im}f_2(z)\geq 0$ for $\mathrm{Im}z\neq 0$ and $f_1(\overline{z})+f_2(\overline{z})=\overline{f_1(z)}+\overline{f_2(z)}=\overline{f_1(z)+f_2(z)}$ we conclude that $f_1(z)+f_2(z)$ is a Nevanlinna function.

Noticing that $f_1(z)>0$ and $f_2(z)>0$ for $z\leq 0$ we arrive at $f_1(z)+f_2(z)>0$ for $z\leq 0$. Thus, $f_1(z)+f_2(z)$ is an S_0-function.

Since

$$\mathrm{Im}\frac{1}{f_j(z)}=\mathrm{Im}\frac{\overline{f_j(z)}}{|f_j(z)|^2}=\frac{-\,\mathrm{Im}f_j(z)}{|f_j(z)|^2} \tag{16}$$

for $j=1,2$ we obtain that

$$\mathrm{Im}\frac{1}{\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}}=\mathrm{Im}\frac{\overline{\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}}}{\left|\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}\right|^2}=\frac{-\,\mathrm{Im}\left(\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}\right)}{\left|\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}\right|^2}.$$

Using (16) we arrive at

$$\mathrm{Im}\frac{1}{\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}}=\frac{\dfrac{\mathrm{Imf_1(z)}}{|f_1(z)|^2}+\dfrac{\mathrm{Imf_2(z)}}{|f_2(z)|^2}}{\left|\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}\right|^2}.$$

Since $\mathrm{Im}z\ \mathrm{Im}f_j(z)>0$ for $\mathrm{Im}z\neq 0$ and $j=1,2$ we arrive at $\mathrm{Im}z\ \mathrm{Im}\dfrac{1}{\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}}\geq 0$ for $\mathrm{Im}z\neq 0$.

Also it is clear that $\dfrac{1}{\dfrac{1}{f_1(\overline{z})}+\dfrac{1}{f_2(\overline{z})}}=\overline{\dfrac{1}{\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}}}$ and $\dfrac{1}{\dfrac{1}{f_1(z)}+\dfrac{1}{f_2(z)}}>0$ for $z\leq 0$. □

Theorem 2.5. *The sequences $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$ and $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$ satisfy the following conditions:*
1) $0<\xi_1\leq\lambda_1\leq\xi_2\leq\lambda_2\leq\cdots\leq\xi_{n_1+n_2+n_3}<\lambda_{n_1+n_2+n_3}$;
2) *If* $\lambda_k=\lambda_{k+1}$, *then* $\xi_k<\lambda_k=\xi_{k+1}=\lambda_{k+1}<\xi_{k+2}$
 $(k=1,2,\ldots,n_1+n_2+n_3-2)$;
 If $\lambda_{n_1+n_2+n_3-1}=\lambda_{n_1+n_2+n_3}$, *then*
 $\xi_{n_1+n_2+n_3-1}<\lambda_{n_1+n_2+n_3-1}=\xi_{n_1+n_2+n_3}=\lambda_{n_1+n_2+n_3}$.
3) *the multiplicity of λ_k and of ξ_k does not exceed 2 for all k.*

Proof. Since $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ are the zeros of $R_{2n_1-1}^{(1)}(z)$ and $\{\mu_k\}_{k=1}^{n_2+n_3}$ are the zeros of the polynomial $\Psi(z)$, we conclude that ξ_ks are the zeros of the polynomial

$$F(z):=R_{2n_1-1}^{(1)}(z)\left(R_{2n_2}^{(2)}(z)R_{2n_3-1}^{(3)}(z)+R_{2n_2-1}^{(2)}(z)R_{2n_3}^{(3)}(z)\right).$$

Consider the ratio

$$\frac{\Phi(z)}{F(z)} = \frac{R^{(1)}_{2n_1}(z)\left(R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z)\right)}{R^{(1)}_{2n_1-1}(z)\left(R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z)\right)}$$
$$+ \frac{R^{(1)}_{2n_1-1}(z)R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3}(z)}{R^{(1)}_{2n_1-1}(z)\left(R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z)\right)}$$
$$= \frac{R^{(1)}_{2n_1}(z)}{R^{(1)}_{2n_1-1}(z)} + \frac{R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3}(z)}{R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z)}$$
$$= \frac{R^{(1)}_{2n_1}(z)}{R^{(1)}_{2n_1-1}(z)} + \frac{1}{\dfrac{1}{\dfrac{R^{(3)}_{2n_3}(z)}{R^{(3)}_{2n_3-1}(z)}} + \dfrac{1}{\dfrac{R^{(2)}_{2n_2}(z)}{R^{(2)}_{2n_2-1}(z)}}}.$$

From [8] we know that $\dfrac{R^{(j)}_{2n_j}(z)}{R^{(j)}_{2n_j-1}(z)}$ is an S_0-function for each $j = 1, 2, 3$. Then

$$\left(\left(\frac{R^{(2)}_{2n_2}(z)}{R^{(2)}_{2n_2-1}(z)}\right)^{-1} + \left(\frac{R^{(3)}_{2n_3}(z)}{R^{(3)}_{2n_3-1}(z)}\right)^{-1}\right)^{-1}$$

is also a S_0-function (see Lemma 2.4.).

So, we conclude that $\dfrac{\Phi(z)}{F(z)}$ is also a S_0-function, and thus zeros of this rational function interlace with its poles as in the statement 1) of the theorem.

Now let us prove statement 3). It is known that the eigenvalues $\{\nu^{(1)}_k\}_{k=1}^{n_1}$ of the Dirichlet-Neumann problem (5)–(7) are simple (see, e.g. [8], Addition II, inequalities (20)), as well as the eigenvalues $\{\mu_k\}_{k=1}^{n_2+n_3}$ of the Dirichlet-Dirichlet problem (8)–(11). This means that the multiplicity of any element of the sequence $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$ does not exceed 2. It was proved in [2] (Theorem 2.2) that the multiplicity of λ_k does not exceed 2. Statement 3) is proved.

Now let's prove statement 2). Let $\lambda_k = \lambda_{k+1}$, then according to statement 1) we have $\lambda_k = \xi_{k+1} = \lambda_{k+1}$. But this means that $R^{(1)}_{2n_1}(\lambda_k) = R^{(2)}_{2n_2}(\lambda_k) = R^{(3)}_{2n_3}(\lambda_k) = 0$ (see the proof of Theorem 2.2 in [2]). It follows from $R^{(1)}_{2n_1}(\lambda_k) = 0$ that $R^{(1)}_{2n_1-1}(\lambda_k) \neq 0$ since the zeros of $R^{(j)}_{2k}(\lambda)$ strictly interlace with the zeros of $R^{(j)}_{2k-1}(\lambda)$ (see [8]). On the other hand, ξ_{k+1} can be double only when $R^{(1)}_{2n_1-1}(\xi_{k+1}) = 0$ and $R^{(2)}_{2n_2}(\xi_{k+1})R^{(3)}_{2n_3-1}(\xi_{k+1}) + R^{(2)}_{2n_2-1}(\xi_{k+1})R^{(3)}_{2n_3-2}(\xi_{k+1}) = 0$. Thus ξ_{k+1} is a simple element of the sequence $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$. Statement 2) is proved.

□

3. Inverse problem

In this section we consider the inverse problem of recovering three strings beads' masses $\{m^{(j)}_k\}_{k=1}^{n_j}$ $(j = 1, 2, 3)$ and intervals' lengths $\{l^{(j)}_k\}_{k=0}^{n_j}$ $(j = 1, 2, 3)$, using the following given data: the total lengths l_1, l_2, l_3 of the strings and the spectra $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$, $\{\nu^{(1)}_k\}_{k=1}^{n_1}$, $\{\mu_k\}_{k=1}^{n_2+n_3}$ of problems (1)–(4), (5)–(7), (8)–(11), respectively.

In the sequel we will need an auxiliary theorem, which is proved in [3], (Theorem 3.1 there), which in our terms has the following form.

Theorem 3.1. *Let l_2 and l_3 be positive numbers. Let three sequences of real numbers $\{\mu_k\}_{k=1}^{n_2+n_3}, \{\tau_k^{(2)}\}_{k=1}^{n_2}, \{\tau_k^{(3)}\}_{k=1}^{n_3}$ be given, which satisfy the following conditions:*

1) $\mu_k < \mu_{k'}$, *if* $k < k'$, $\quad \tau_k^{(j)} < \tau_{k'}^{(j)}$, *if* $k < k'$ *and* $j = 2, 3$.

2)

$$\{\mu_k\}_{k=1}^{n_2+n_3} \cap \{\tau_k^{(2)}\}_{k=1}^{n_2} = \emptyset,$$

$$\{\mu_k\}_{k=1}^{n_2+n_3} \cap \{\tau_k^{(3)}\}_{k=1}^{n_3} = \emptyset,$$

$$\{\tau_k^{(2)}\}_{k=1}^{n_2} \cap \{\tau_k^{(3)}\}_{k=1}^{n_3} = \emptyset.$$

3) *Elements of the sequence $\{\chi_k\}_{k=1}^{n_2+n_3} := \{\tau_k^{(2)}\}_{k=1}^{n_2} \cup \{\tau_k^{(3)}\}_{k=1}^{n_3}$ indexed such that $\chi_k < \chi_{k'}$, if $k < k'$ interlace with elements of the sequence $\{\mu_k\}_{k=1}^{n_2+n_3}$, i.e.*

$$0 < \mu_1 < \chi_1 < \mu_2 < \chi_2 < \cdots < \mu_{n_2+n_3} < \chi_{n_2+n_3}.$$

Then there is a unique collection of sequences $\{m_k^{(2)}\}_{k=1}^{n_2}, \{m_k^{(3)}\}_{k=1}^{n_3}$, $\{l_k^{(2)}\}_{k=0}^{n_2}$, $\{l_k^{(3)}\}_{k=0}^{n_3}$ such that $\sum\limits_{k=0}^{n_j} l_k^{(j)} = l_j \quad (j = 2, 3)$, which generates problem (8)–(11) with the spectrum $\{\mu_k\}_{k=1}^{n_2+n_3}$ and the problem

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)} z u_k^{(j)} = 0 \quad (k = 1, 2, \ldots, n_j), \tag{17}$$

$$u_0^{(j)} = u_{n_j+1}^{(j)} = 0 \tag{18}$$

with $j = 2$, which has the spectrum $\{\tau_k^{(2)}\}_{k=1}^{n_2}$ and problem (17),(18) with $j = 3$, which has the spectrum $\{\tau_k^{(3)}\}_{k=1}^{n_3}$.

Now we present the main result.

Theorem 3.2. *Let three positive numbers l_1, l_2, l_3 be given together with sequences of positive numbers $\{\nu_k^{(1)}\}_{k=1}^{n_1}, \{\mu_k\}_{k=1}^{n_2+n_3}, \{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$ which satisfy the following conditions:*

$$1) \quad 0 < \xi_1 < \lambda_1 < \xi_2 < \lambda_2 < \cdots < \xi_{n_1+n_2+n_3} < \lambda_{n_1+n_2+n_3}, \tag{19}$$

where $\{\xi_k\}_{k=1}^{n_1+n_2+n_3} = \{\nu_k^{(1)}\}_{k=1}^{n_1} \bigcup \{\mu_k\}_{k=1}^{n_2+n_3}$;

$$2) \quad (-1)^{n_1} \left(\prod_{j=1}^{n_1} \frac{l_1}{\nu_0^{(1)} - \nu_j^{(1)}} + \sum_{k=1}^{n_1} \frac{P_3(\nu_k^{(1)})}{P_2(\nu_k^{(1)})} \prod_{j=0, j\neq k}^{n_1} \frac{1}{\nu_k^{(1)} - \nu_j^{(1)}} \right) > 0; \tag{20}$$

$$3) \quad (-1)^{n_2+n_3} \left(\prod_{j=1}^{n_2+n_3} \frac{l_2 l_3}{\mu_0 - \mu_j} + \sum_{k=1}^{n_2+n_3} \frac{P_3(\mu_k)}{P_1(\mu_k)} \prod_{j=0, j\neq k}^{n_2+n_3} \frac{1}{\mu_k - \mu_j} \right) > 0. \tag{21}$$

Then:

1) there exist sequences of positive numbers $\{m_k^{(j)}\}_{k=1}^{n_j}$ $\quad (j = 1, 2, 3)$ and $\{l_k^{(j)}\}_{k=0}^{n_j}$ $\quad (j = 1, 2, 3)$, which generate problems (1)–(4), (5)–(7), (8)–(11) such that $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$ is the spectrum of problem (1)–(4), $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ is the spectrum of problem (5)–(7), $\{\mu_k\}_{k=1}^{n_2+n_3}$ is the spectrum of problem (8)–(11), and total length of strings are l_1, l_2, l_3, respectively;
2) these data uniquely determine the masses and the intervals on the first string.

Proof. Let us construct the following polynomials:

$$P_1(z) = \prod_{k=1}^{n_1}\left(1 - \frac{z}{\nu_k^{(1)}}\right),$$

$$P_2(z) = (l_2 + l_3)\prod_{k=1}^{n_2+n_3}\left(1 - \frac{z}{\mu_k}\right),$$

$$P_3(z) = (l_1 l_2 + l_1 l_3 + l_2 l_3)\prod_{k=1}^{n_1+n_2+n_3}\left(1 - \frac{z}{\lambda_k}\right)$$

and consider the functional equation

$$P_3(z) = P_1(z)Y(z) + P_2(z)X(z), \tag{22}$$

where $X(z), Y(z)$ are unknown polynomials.

Substituting $z = \nu_k^{(1)}$ into (22) we obtain

$$X(\nu_k^{(1)}) = \frac{P_3(\nu_k^{(1)})}{P_2(\nu_k^{(1)})}, \quad k = 1, 2, \ldots, n_1. \tag{23}$$

The condition of strict interlacing (19) implies that the denominator $P_2(\nu_k^{(1)}) \neq 0$.

Let us find the polynomial $X(z)$, using its known values (23) and setting by definition $\nu_0^{(1)} = 0$ and $X(\nu_0^{(1)}) = X(0) = l_1$.

We construct the following Lagrange interpolating polynomial

$$X(z) = \sum_{k=0}^{n_1} X(\nu_k^{(1)}) \prod_{j=0, j\neq k}^{n_1} \frac{z - \nu_j^{(1)}}{\nu_k^{(1)} - \nu_j^{(1)}}.$$

Denote by $\{\zeta_k^{(1)}\}_{k=1}^{n_1}$ the zeros $X(z)$. We need to show that they interlace with the elements of the sequence $\{\nu_k^{(1)}\}_{k=1}^{n_1}$.

To this end, let us find the signs of the values attained by the polynomial $X(z)$ at the points $z = \nu_k^{(1)}$ $(k = 1, 2, \ldots)$. By definition $\{\xi_k\}_{k=1}^{n_1+n_2+n_3} = \{\nu_k^{(1)}\}_{k=1}^{n_1} \bigcup \{\mu_k\}_{k=1}^{n_2+n_3}$, and therefore $\nu_k^{(1)}$ must coincide with some element of the sequence $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$. Let $\nu_k^{(1)} = \xi_p$, where $p \geq k \geq 1$. So, $P_3(\nu_k^{(1)}) = P_3(\xi_p)$ and due to (19) we have $P_3(\nu_k^{(1)})(-1)^{p-1} = P_3(\xi_p)(-1)^{p+1} > 0$. It follows from $\nu_k^{(1)} = \xi_p$ that there must be $p - k$ elements of the set $\{\mu_k\}_{k=1}^{n_2+n_3}$ on the interval $(-\infty, \nu_k^{(1)})$. This means that $(-1)^{p-k} P_2(\nu_k^{(1)}) > 0$. Thus, (23) implies $(-1)^{k+1} X(\nu_k^{(1)}) > 0$, for all $k \geq 1$. Also we notice that (20) implies $(-1)^{n_1} X(z) > 0$ when $z \to \infty$.

So, we have

$$0 < \nu_1^{(1)} < \zeta_1^{(1)} < \nu_2^{(1)} < \zeta_2^{(1)} < \cdots < \nu_{n_1}^{(1)} < \zeta_{n_1}^{(1)}. \tag{24}$$

Due to inequalities (24), the sequences $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ and $\{\zeta_k^{(1)}\}_{k=1}^{n_1}$ can be considered as the spectra of the Dirichlet-Neumann and Dirichlet-Dirichlet problems. Using these spectra and the length l_1 we can construct a Stieltjes string, that is, to find the masses $\{m_k^{(1)}\}_{k=1}^{n_1}$ and the lengths $\{l_k^{(1)}\}_{k=0}^{n_1}$ by the known procedure described in [8].

Due to (24) the ratio $\dfrac{X(z)}{P_1(z)}$ is an S_0-function and it can be expanded in the continued fraction

$$\frac{X(z)}{P_1(z)} = a_{n_1}^{(1)} + \cfrac{1}{-b_{n_1}^{(1)} z + \cfrac{1}{a_{n_1-1}^{(1)} + \cdots + \cfrac{1}{-b_1^{(1)} z + \cfrac{1}{a_0^{(1)}}}}},$$

where $a_k^{(1)} > 0 \quad (k = 0, 1, \ldots, n_1)$, $b_k^{(1)} > 0 \quad (k = 1, 2, \ldots, n_1)$.

We identify these coefficients with the masses of beads and the lengths of the intervals between them on the first string, i.e. $a_k = l_k^{(1)}$, $(k = 0, 1, \ldots, n_1)$, $b_k = m_k^{(1)}$, $(k = 1, 2, \ldots, n_1)$. This means that the sequences $\{l_k^{(1)}\}_{k=0}^{n_1}$, $\{m_k^{(1)}\}_{k=1}^{n_1}$ generate problem (5)–(7) with the spectrum $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ and problem (17),(18) with the spectrum $\{\zeta_k^{(1)}\}_{k=1}^{n_1}$. Then

$$R_{2n_1-1}^{(1)}(z) = P_1(z) \tag{25}$$

and

$$R_{2n_1}^{(1)}(z) = X(z). \tag{26}$$

Let us find $Y(z)$. To this end, we substitute $z = \mu_k$ into equation (22) and obtain

$$Y(\mu_k) = \frac{P_3(\mu_k)}{P_1(\mu_k)}, \quad k = 1, 2, \ldots, n_2 + n_3. \tag{27}$$

It follows from the condition of strict interlacing (19) that $P_1(\mu_k) \neq 0$. We construct the polynomial $Y(z)$, using its known values (27) and setting by definition $\mu_0 = 0$ and $Y(\mu_0) = Y(0) = l_2 l_3$. The Lagrange interpolating polynomial has the form

$$Y(z) = \sum_{k=0}^{n_2+n_3} Y(\mu_k) \prod_{j=0, j\neq k}^{n_2+n_3} \frac{z - \mu_j}{\mu_k - \mu_j}.$$

Denote by $\{\chi_k\}_{k=1}^{n_2+n_3}$ the zeros of the polynomial $Y(z)$. We need to show that they interlace with the elements of the sequence $\{\mu_k\}_{k=1}^{n_2+n_3}$. To this end, we find the signs of values attained by the polynomial $Y(z)$ at points $z = \mu_k \quad (k = 1, 2, \ldots, n_2 + n_3)$. Since $\{\xi_k\}_{k=1}^{n_1+n_2+n_3} = \{\nu_k^{(1)}\}_{k=1}^{n_1} \bigcup \{\mu_k\}_{k=1}^{n_2+n_3}$, each μ_k must coincide with some element of the sequence $\{\xi_k\}_{k=1}^{n_1+n_2+n_3}$. Let $\mu_k = \xi_p$, where $p \geq k \geq 1$. Thus, $P_3(\mu_k) = P_3(\xi_p)$ and due to (19) we have $P_3(\mu_k)(-1)^{p-1} = P_3(\xi_p)(-1)^{p+1} > 0$. From $\mu_k = \xi_p$ it follows that there must be $p - k$ elements of the sequence $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ on the interval $(-\infty, \mu_k)$. This means that $(-1)^{p-k} P_1(\mu_k) > 0$. Now it follows from equation (27) that $(-1)^{k+1} Y(\mu_k) > 0$ for $k \geq 1$. Finally, (21) implies $(-1)^{n_2+n_3} Y(z) > 0$ for $z \to \infty$.

Thus, we have

$$0 < \mu_1 < \chi_1 < \mu_2 < \chi_2 < \cdots < \mu_{n_2+n_3} < \chi_{n_2+n_3}.$$

The sequence $\{\mu_k\}_{k=1}^{n_2+n_3}$ can be considered as the spectrum of the Dirichlet-Dirichlet problem on the interval formed by the union of the second and the third strings, while $\{\chi_k\}_{k=1}^{n_2+n_3}$ as the union of the spectra of the Dirichlet-Dirichlet problems on the second and the third strings.

We have the sequence $\{\mu_k\}_{k=1}^{n_2+n_3}$ and the sequence $\{\chi_k\}_{k=1}^{n_2+n_3}$, which we consider as a union of sequences $\{\tau_k^{(2)}\}_{k=1}^{n_2}$, $\{\tau_k^{(3)}\}_{k=1}^{n_3}$. Of course, we can arbitrarily identify them. Obviously, $\{\mu_k\}_{k=1}^{n_2+n_3}$, $\{\tau_k^{(2)}\}_{k=1}^{n_2}$, $\{\tau_k^{(3)}\}_{k=1}^{n_3}$ satisfy the conditions of Theorem 3.1. Thus,

$\{\mu_k\}_{k=1}^{n_2+n_3}$ is the spectrum of problem (8)–(11), $\{\tau_k^{(2)}\}_{k=1}^{n_2}$ is the spectrum of problem (17), (18) with $j=2$ and $\{\tau_k^{(3)}\}_{k=1}^{n_2}$ is the spectrum of problem (17), (18) with $j=3$.

The method of finding $\{m_k^{(2)}\}_{k=1}^{n_2}, \{m_k^{(3)}\}_{k=1}^{n_3}, \{l_k^{(2)}\}_{k=0}^{n_2}, \{l_k^{(3)}\}_{k=0}^{n_3}$ can be found in [3]. It is as follows. Let us construct the following polynomials:

$$Q_j(z) = l_j \prod_{k=1}^{n_j}\left(1-\frac{z}{\tau_k^{(j)}}\right), \quad j=2,3,$$

and the Lagrange interpolating polynomials

$$R_2(z) = \sum_{k=1}^{n_j} \frac{zP_2(\tau_k^{(2)})}{\tau_k^{(2)}Q_3(\tau_k^{(2)})} \prod_{p=1,p\neq k}^{n_2} \frac{z-\tau_p^{(2)}}{\tau_k^{(2)}-\tau_p^{(2)}} + \prod_{k=1}^{n_2}\frac{\tau_k^{(2)}-z}{\tau_k^{(2)}},$$

$$R_3(z) = \sum_{k=1}^{n_3} \frac{zP_2(\tau_k^{(3)})}{\tau_k^{(3)}Q_2(\tau_k^{(3)})} \prod_{p=1,p\neq k}^{n_3} \frac{z-\tau_p^{(3)}}{\tau_k^{(3)}-\tau_p^{(3)}} + \prod_{k=1}^{n_3}\frac{\tau_k^{(3)}-z}{\tau_k^{(3)}}.$$

Denote by $\{\alpha_k^{(j)}\}_{k=1}^{n_j}$, $(j=2,3)$ the zeros of the polynomial $R_j(z)$ indexed in increasing order.

According to the proof of Theorem 2.1 in [3] the following interlacing conditions are fulfilled:

$$0<\alpha_1^{(j)}<\tau_1^{(j)}<\alpha_2^{(j)}<\tau_2^{(j)}<\cdots<\alpha_{n_j}^{(j)}<\tau_{n_j}^{(j)}, \quad j=2,3.$$

Due to these inequalities we conclude that $\dfrac{Q_j(z)}{R_j(z)}$ is an S_0 - function for $j=2,3$. Expanding the ratio $\dfrac{Q_j(z)}{R_j(z)}$ $(j=2,3)$ into a continued fraction we get

$$\frac{Q_j(z)}{R_j(z)} = a_{n_j}^{(j)} + \cfrac{1}{-b_{n_j}^{(j)}z + \cfrac{1}{a_{n_j-1}^{(j)} + \cdots + \cfrac{1}{-b_1^{(j)}z + \cfrac{1}{a_0^{(j)}}}}},$$

where all $a_k^{(j)}$ and $b_k^{(j)}$ are positive numbers. We identify them as $a_k^{(j)} = l_k^{(j)}, b_k^{(j)} = m_k^{(j)}$, that is, we assume that the numbers $\{m_k^{(j)}\}_{k=1}^{n_j}$, $(j=2,3)$ are the masses of the beads of j-th Stieltjes string and $\{l_k^{(j)}\}_{k=0}^{n_j}$, $(j=2,3)$ are the lengths of the subintervals of the j-th Stieltjes string. Obviously, we have

$$\sum_{k=0}^{n_j} l_k^{(j)} = \frac{Q_j(0)}{R_j(0)} = l_j.$$

Then, according to [3], $\{\mu_k\}_{k=1}^{n_2+n_3}$ is the spectrum of problem (8)–(11) and $\{\tau_k^{(j)}\}_{k=1}^{n_j}$ $(j=2,3)$ are the spectra of problems (17), (18) generated by these masses and lengths.

Thus,

$$R_{2n_2-1}^{(2)}(z)R_{2n_3}^{(3)}(z) + R_{2n_3-1}^{(3)}(z)R_{2n_2}^{(2)}(z) = P_2(z) \tag{28}$$

and

$$R_{2n_j}^{(j)}(z) = Q_j(z) \quad (j=2,3), \tag{29}$$

$$R_{2n_j-1}^{(j)}(z) = R_j(z) \quad (j=2,3). \tag{30}$$

Now let us prove that the spectrum of problem (1)–(4) generated by the obtained masses and lengths coincides with $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$.

We consider equation (22) under the conditions $Y(0) = l_2 l_3$, $X(0) = l_1$. We have already found its solution and now we need to prove the uniqueness of it.

Assume that there is another solution $(\tilde{X}(z); \tilde{Y}(z))$, that is

$$P_3(z) = P_1(z)\tilde{Y}(z) + P_2(z)\tilde{X}(z). \tag{31}$$

Subtracting (31) from (22) we obtain

$$P_1(z)(Y(z) - \tilde{Y}(z)) = -P_2(z)(X(z) - \tilde{X}(z)).$$

From this we see that $Y(z) - \tilde{Y}(z) = CP_2(z)$ and $X(z) - \tilde{X}(z) = -CP_1(z)$, where C is a constant.

Since $\tilde{Y}(0) = Y(0) = l_2 l_3$ and $\tilde{X}(0) = X(0) = l_1$, we conclude that $C = 0$. Thus, we have proved the uniqueness of the solution of (31) under the conditions $\tilde{Y}(0) = Y(0) = l_2 l_3$ and $\tilde{X}(0) = X(0) = l_1$,.

The spectrum of problem (1)–(4) with the obtained masses and intervals according to (14) is the sequence of the zeros of the polynomial

$$R^{(1)}_{2n_1-1}(z)R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3}(z) + +R^{(1)}_{2n_1}(z)\left(R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z)\right).$$

According to (25),(26),(28),(29),(30) we have

$$\begin{aligned} R^{(1)}_{2n_1-1}(z)R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3}(z) + +R^{(1)}_{2n_1}(z)&\left(R^{(2)}_{2n_2}(z)R^{(3)}_{2n_3-1}(z) + R^{(2)}_{2n_2-1}(z)R^{(3)}_{2n_3}(z)\right) \\ &= P_1(z)Y(z) + P_2(z)X(z) = P_3(z) \end{aligned}$$

and the sequence of the zeros $P_3(z)$ is $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}$. □

Remark. It is clear from the proof of Theorem 3.2 that the data $\{\nu_k^{(1)}\}_{k=1}^{n_1}$, $\{\mu_k\}_{k=1}^{n_2+n_3}$, $\{\lambda_k\}_{k=1}^{n_1+n_2+n_3}, l_1, l_2, l_3$ uniquely determine the masses $\{m_k^{(1)}\}$ and the lengths $\{l_k^{(1)}\}$ on the first string, but because of arbitrary choice of choosing $\{\tau_k^{(2)}\}$ and $\{\tau_k^{(3)}\}$ among $\{\chi_k\}_{k=1}^{n_2+n_3}$, the masses $\{m_k^{(j)}\}_{k=1}^{n_j}$ $(j = 2, 3)$ and the lengths $\{l_k^{(j)}\}_{k=1}^{n_j}$ $(j = 2, 3)$ are not determined uniquely.

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South-Ukrainian National Pedagogical University, named after K. D. Ushynsky, Staro-portofrankovskaya str. 26, Odesa, Ukraine, 65020
E-mail address: `nastysha00301@gmail.com`

South-Ukrainian National Pedagogical University, named after K. D. Ushynsky, Staro-portofrankovskaya str. 26, Odesa, Ukraine, 65020
E-mail address: `vpivovarchik@gmail.com`

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