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ASYMPTOTIC REPRESENTATIONS OF REGULARLY VARYING $P_\omega(Y_0, Y_1, \lambda_0)$-SOLUTIONS OF A DIFFERENTIAL EQUATION OF THE SECOND ORDER CONTAINING THE PRODUCT OF DIFFERENT TYPES OF NONLINEARITIES OF THE UNKNOWN FUNCTION AND ITS DERIVATIVE

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We establish necessary and sufficient conditions for the existence of regularly varying solutions of the second-order differential equations whose right-hand sides contain the product of a regularly varying nonlinearity of the unknown function and a rapidly varying nonlinearity of the derivative of the unknown function as the arguments tend either to zero or to infinity. Asymptotic representations of these solutions and their first-order derivatives are also found.

1. Statement of the Problem

Consider a differential equation

$$y'' = \alpha_0 p(t)\varphi_1(y)\varphi_0(y'), \tag{1.1}$$

where $\alpha_0 \in \{-1; 1\}$, $p\colon [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ is a semicontinuous function, $\varphi_1\colon \Delta_{Y_0} \to]0, +\infty[$ is a regularly varying [1, p. 17] function of order σ_1 as the argument tends to Y_0, and $\varphi_0\colon \Delta_{Y_1} \to]0, +\infty[$ is a function twice continuously differentiable on Δ_{Y_1} and such that

$$\lim_{\substack{y\to Y_1\\ y\in\Delta_{Y_1}}} \varphi_0(y) \in \{0, +\infty\}, \quad \varphi_0'(y) \ne 0 \quad \text{for} \quad y \in \Delta_{Y_1}, \quad \lim_{\substack{y\to Y_1\\ y\in\Delta_{Y_1}}} \frac{\varphi_0(y)\varphi_0''(y)}{\left(\varphi_0'(y)\right)^2} = 1. \tag{1.2}$$

Moreover, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ or $]Y_i, y_i^0]$, $i \in \{0, 1\}$. For $Y_i = +\infty$ $(Y_i = -\infty)$, we assume that $y_i^0 > 0$ $(y_i^0 < 0)$.

Definition 1.1. *A solution y of Eq. (1.1) defined on $[t_0, \omega[\subset [a, \omega[$ is called a $P_\omega(Y_0, Y_1, \lambda_0)$-solution $-\infty \le \lambda_0 \le +\infty$ if*

$$y^{(i)}\colon [t_0, \omega[\longrightarrow \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i, \quad i = 0, 1, \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

The aim of the present paper is to establish necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of Eq. (1.1) and asymptotic representations of these solutions and their first-order derivatives as $t \uparrow \omega$ for $\lambda_0 \in \mathbb{R}\setminus\{0, 1\}$.

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According to the *a priori* asymptotic properties, these solutions are regularly varying functions of order $\dfrac{\lambda_0}{\lambda_0 - 1}$ and their first-order derivatives are regularly varying functions of order $\dfrac{1}{\lambda_0 - 1}$ as $t \uparrow \omega$ [2].

2. Main Notation and Definitions

We introduce the following notation necessary in what follows:

$$\pi_\omega(t) = \begin{cases} t & \text{for} \quad \omega = +\infty, \\ t - \omega & \text{for} \quad \omega < +\infty, \end{cases} \qquad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi_0(z) = \int_{A_\omega}^{z} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)}, \qquad \Phi_1(z) = \int_{A_\omega}^{z} \Phi_0(s)\,ds,$$

$$A_\omega = \begin{cases} y_1^0 & \text{for} \quad \displaystyle\int_{y_1^0}^{Y_1} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)} = \pm\infty, \\ Y_1 & \text{for} \quad \displaystyle\int_{y_1^0}^{Y_1} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)} = \text{const}, \end{cases}$$

$$Z_0 = \lim_{\substack{z \to Y_1 \\ z \in \Delta_{Y_1}}} \Phi_0(z), \qquad Z_1 = \lim_{\substack{z \to Y_1 \\ z \in \Delta_{Y_1}}} \Phi_1(z),$$

$$F(t) = \frac{\pi_\omega(t) I_1'(t)}{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}.$$

In the case where

$$y_1^0 \lim_{t \uparrow \omega} |\pi_\omega(\tau)|^{\frac{1}{\lambda_0 - 1}} = Y_1,$$

we denote

$$I(t) = \alpha_0 y_0^0 \left|\frac{\lambda_0 - 1}{\lambda_0}\right|^{\sigma_1} \int_{B_\omega^0}^{t} |\pi_\omega(\tau)|^{\sigma_1} p(\tau) \theta_1\left(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0 - 1}} y_0^0\right) d\tau,$$

$$B_\omega^0 = \begin{cases} b & \text{for} \quad \displaystyle\int_b^{\omega} |\pi_\omega(\tau)|^{\sigma_1} p(\tau)\theta_1\left(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0 - 1}} y_0^0\right) d\tau = +\infty, \\ \omega & \text{for} \quad \displaystyle\int_b^{\omega} |\pi_\omega(\tau)|^{\sigma_1} p(\tau)\theta_1\left(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0 - 1}} y_0^0\right) d\tau < +\infty, \end{cases}$$

$$I_1(t) = \int\limits_{B_\omega^1}^{t} \frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau, \quad B_\omega^1 = \begin{cases} b & \text{for} \quad \displaystyle\int_b^\omega \frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}d\tau = \pm\infty, \\ \omega & \text{for} \quad \displaystyle\int_b^\omega \frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}d\tau < +\infty, \end{cases}$$

where $b \in [a;\omega[$ is chosen such that

$$y_1^0 \lim_{t\uparrow\omega} |\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}} \in \Delta_{Y_1}$$

for $t \in [b;\omega]$.

Remark 2.1. In view of conditions (1.2) imposed on the function φ_0, we conclude that $Z_0, Z_1 \in \{0, +\infty\}$ and

$$\lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_1}}} \frac{\Phi_0''(z)\Phi_0(z)}{(\Phi_0'(z))^2} = 1, \quad \lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_1}}} \frac{\Phi_1''(z)\Phi_1(z)}{(\Phi_1'(z))^2} = 1. \tag{2.1}$$

Remark 2.2. The following statements are true:

(i) $\Phi_0(z) = (\sigma_1 - 1)\,\dfrac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(z)}{\varphi_0'(z)}\,[1+o(1)]$ as $z \to Y_1$ $(y \in \Delta_{Y_1})$. This yields

$$\operatorname{sign}\left(\varphi_0'(z)\Phi_0(z)\right) = \operatorname{sign}(\sigma_1 - 1) \quad \text{for} \quad z \in \Delta_{Y_1}.$$

(ii) $\Phi_1(z) = \dfrac{\Phi_0^2(z)}{y\Phi_0'(z)}\,[1+o(1)]$ as $z \to Y_1$ $z \in \Delta_{Y_1}$. This yields

$$\operatorname{sign}(\Phi_1(z)) = y_0^0 \quad \text{for} \quad z \in \Delta_{Y_1}.$$

(iii) The functions Φ_0^{-1} and Φ_1^{-1} exist and are slowly varying as inverse functions for rapidly varying functions as the arguments tend to Y_1.

(iv) The function $\Phi_1'\left(\Phi_1^{-1}\right)$ is a regularly varying function of order 1 as the argument tends to Y_1. Indeed, in view of (2.1), the following relation is true:

$$\lim_{z\to Z_1} \frac{\left(\Phi_1'\left(\Phi_1^{-1}(z)\right)\right)' z}{\Phi_1'\left(\Phi_1^{-1}(z)\right)} = \lim_{z\to Z_1} \frac{\Phi_1''\left(\Phi_1^{-1}(z)\right) z}{\left(\Phi_1'\left(\Phi_1^{-1}(z)\right)\right)^2}$$

$$= \lim_{y\to Y_1} \frac{\Phi_1''\left(\Phi_1^{-1}(\Phi_1(y))\right)\Phi_1(y)}{\left(\Phi_1'\left(\Phi_1^{-1}(\Phi_1(y))\right)\right)^2} = \lim_{y\to Y_1} \frac{\Phi_1''(y)\Phi_1(y)}{\left(\Phi_1'(y)\right)^2} = 1.$$

(v) The function $\Phi_1^{-1}(z)\,\dfrac{\Phi_1'\left(\Phi_1^{-1}(z)\right)}{z}$ is slowly varying as $z \to Z_1$.

Definition 2.1. *Let $Y \in \{0, \pm\infty\}$ and let Δ_Y be a certain one-sided neighborhood of Y. A continuously differentiable function $L\colon \Delta_Y \to]0; +\infty[$ is called a normalized slowly varying function as $y \to Y$, $y \in \Delta_Y$ [2, pp. 2–3] if*

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}} \frac{yL'(y)}{L(y)} = 0.$$

Definition 2.2. *A slowly varying, as $y \to Y$, $y \in \Delta_Y$, function $\theta\colon \Delta_Y \to]0; +\infty[$ satisfies the condition S as the argument tends to Y (see, e.g., [2]) if, for any normalized function $L\colon \Delta_Y \to]0; +\infty[$ slowly varying as $y \to Y$, $y \in \Delta_Y$, the following relation is true:*

$$\theta(yL(y)) = \theta(y)(1 + o(1)) \quad as \quad y \to Y, \quad y \in \Delta_Y.$$

Condition S is satisfied by functions of the form $\ln|y|$, $|\ln|y||^{\mu}$, $\mu \in R$, $\ln\ln|y|$, and many other functions.

3. Main Results

We prove the following theorem:

Theorem 3.1. *Suppose that $\sigma_1 \in R\backslash\{1\}$ and a function θ_1 satisfies the condition S. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, where $\lambda_0 \in \mathbb{R}\backslash\{0, 1\}$, of Eq. (1.1), it is necessary and if the following condition is satisfied:*

$$(2\lambda_0 - 1 + \sigma_1)(\lambda_0 - 1) < 0 \quad for \quad t \in]b, \omega[\tag{3.1}$$

and a finite or infinite limit

$$\frac{\sqrt{\left|\dfrac{\pi_\omega(t) I_1'(t)}{I_1(t)}\right|}}{\ln|I_1(t)|} \tag{3.2}$$

exists, then it is also sufficient that the following conditions be satisfied:

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad y_1^0 \alpha_0(\lambda_0 - 1)\pi_\omega(t) > 0 \quad for \quad t \in [a; \omega[, \tag{3.3}$$

$$y_0^0 \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0 - 1}} = Y_0, \quad y_1^0 \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1, \quad \lim_{t\uparrow\omega} I_1(t) = Z_1, \tag{3.4}$$

$$\lim_{t\uparrow\omega} \frac{I_1''(t) I_1(t)}{(I_1'(t))^2} = 1, \tag{3.5}$$

$$\lim_{t\uparrow\omega} \frac{I'(t) I_1(t)}{I_1'(t) I(t)} = 1, \quad \lim_{t\uparrow\omega} \frac{\Phi_0(\Phi_1^{-1}(I_1(t)))}{I(t)} = 1, \quad \lim_{t\uparrow\omega} F(t) = \frac{1}{\lambda_0 - 1}. \tag{3.6}$$

Moreover, for each solution of this kind, the following asymptotic representations are true as $t \uparrow \omega$:

$$y'(t) = \Phi_1^{-1}(I_1(t))[1+o(1)], \quad y(t) = \frac{(\lambda_0 - 1)\Phi_1^{-1}(I_1(t))\pi_\omega(t)}{\lambda_0}[1+o(1)]. \tag{3.7}$$

Proof. *Necessity.* Let a function $y\colon [t_0,\omega[\to \Delta_{Y_0}$ be a $P_\omega(Y_0, Y_1, \lambda_0)$-solution of Eq. (1.1) for which $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$. Then, according to properties of these solutions established by Evtukhov (see, e.g., [2]), we get

$$\frac{y'(t)}{y(t)} = \frac{\lambda_0}{(\lambda_0 - 1)\pi_\omega(t)}[1+o(1)], \quad \frac{y''(t)}{y'(t)} = \frac{1}{(\lambda_0 - 1)\pi_\omega(t)}[1+o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.8}$$

This yields (3.3).

Relation (3.8) also implies that $y(t)$ is a regularly varying function of order $\dfrac{\lambda_0}{\lambda_0 - 1}$ as $t \uparrow \omega$. Hence [3], it can be represented in the form

$$y(t) = |\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0 - 1}} L_1(t) \quad \text{as} \quad t \uparrow \omega, \tag{3.9}$$

where $L_1\colon \Delta_{Y_0} \to]0,+\infty[$ is a function slowly varying as $t \uparrow \omega$. Thus, by using properties of regularly varying functions [3], we establish the first condition in (3.4). In a similar way, we show that the second condition in (3.4) is also satisfied.

As $t \uparrow \omega$, it follows from (1.1) and (3.8) that

$$\frac{y''}{\varphi_0(y')} = \alpha_0 p(t)\theta_1(y(t))|y(t)|^{\sigma_1} p(t)[1+o(1)],$$

and, in view of (3.9), we get

$$\frac{y''}{\varphi_0(y')} = \alpha_0 p(t)\theta_1\left(|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0 - 1}} L_1(t)\right)|y(t)|^{\sigma_1}[1+o(1)]. \tag{3.10}$$

Since the function L_1 in (3.9) is slowly varying as the argument tends to Y_1, according to the condition S satisfied for the function θ_1, in view of relation (3.10) as $t \uparrow \omega$, we conclude that

$$\frac{y''}{\varphi_0(y')} = \alpha_0 p(t)\theta_1\left(|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0 - 1}}\right)|y(t)|^{\sigma_1}[1+o(1)]. \tag{3.11}$$

Taking into account the first condition in (3.8) as $t \uparrow \omega$, we obtain

$$\frac{y''}{\varphi_0(y')|y'(t)|^{\sigma_1}} = \alpha_0 p(t)\theta_1\left(|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0 - 1}}\right)\left|\frac{(\lambda_0 - 1)\pi_\omega(t)}{\lambda_0}\right|^{\sigma_1}[1+o(1)]. \tag{3.12}$$

Integrating this relation from t_0 to t, as $t \uparrow \omega$, we find

$$\int_{y'(t_0)}^{y'(t)} \frac{dz}{\varphi_0(z)|z|^{\sigma_1}} = \alpha_0 \left|\frac{(\lambda_0 - 1)}{\lambda_0}\right|^{\sigma_1} \int_{t_0}^{t} p(\tau)\theta_1\left(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0 - 1}}\right)|\pi_\omega(\tau)|^{\sigma_1}[1+o(1)]\,d\tau.$$

By using the last equality, in view of the fact that $y' \to Y_1$, $Y_1 \in \Delta_{Y_1}$, and the choice of A_ω, we conclude that

$$\Phi_0(y'(t)) = I(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.13}$$

By the properties of the function Φ_0, we get

$$y'(t) = \Phi_0^{-1}(I(t))[1 + o(1)] \quad \text{as} \quad t \uparrow \omega.$$

It follows from this relation and (3.8) that

$$y''(t) = \frac{\Phi_0^{-1}(I(t))}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.14}$$

Further, in view of (3.13) and (3.14), we get

$$y''\Phi_0(y') = \frac{I(t)\Phi_0^{-1}(I(t))}{(\lambda_0 - 1)\pi_\omega(t)}[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.15}$$

As in the case of relation (3.13), we can write

$$\Phi_1(y'(t)) = I_1(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.16}$$

In view of the properties of the function Φ_1, we obtain

$$y'(t) = \Phi_1^{-1}(I_1(t))[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.17}$$

Thus, the first relation in (3.7) and the third condition in (3.4) are true. By virtue of (3.8) and (3.17), we obtain the second relation in (3.7).

Further, by using (3.12) and (3.13), we get

$$\frac{y''(t)\Phi_0'(y'(t))}{\Phi_0(y'(t))} = \frac{I'(t)}{I(t)} [1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.18}$$

It follows from (3.8) and (3.18) that

$$\frac{y''(t)\pi_\omega(t)}{y'(t)} \frac{y'(t)\Phi_0'(y'(t))}{\Phi_0(y'(t))} = \frac{\pi_\omega(t)I'(t)}{I(t)}[1 + o(1)] \quad \text{as} \quad t \uparrow \omega.$$

Since the function Φ_0 satisfies Remarks 2.1 and 2.2, we get

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \pm\infty.$$

Similarly, by using (3.15) and (3.16), we obtain

$$\frac{y''(t)\Phi_1'(y'(t))}{\Phi_1(y'(t))} = \frac{I_1'(t)}{I_1(t)} [1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{3.19}$$

This yields

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\pm\infty,$$

and

$$\lim_{t\uparrow\omega}\frac{I_1''(t)I_1(t)}{(I_1'(t))^2}=1.$$

Hence, condition (3.5) is satisfied.

By using (3.18) and (3.19), we find

$$\lim_{t\uparrow\omega}\frac{I'(t)I_1(t)}{I_1'(t)I(t)}=\lim_{t\uparrow\omega}\frac{\Phi_0'(y'(t))\Phi_1(y'(t))}{\Phi_0(y'(t))\Phi_1'(y'(t))}=\lim_{z\to Y_1}\frac{\Phi_1''(z)\Phi_1(z)}{\left(\Phi_1'(z)\right)^2}=1,$$

i.e., the first condition of the theorem in (3.6) is satisfied.

Since $\Phi_1'\left(\Phi_1^{-1}\right)$ is a regularly varying function of order 1 as the argument tends to Y_1 [see Remark 2.2 and (3.13)], the second condition of the theorem in (3.6) is satisfied.

By using (3.8) and (3.19), we get

$$\frac{y''(t)\pi_\omega(t)}{y'(t)}\,\frac{y'(t)\Phi_1'(y'(t))}{\Phi_1(y'(t))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}[1+o(1)]\quad\text{as}\quad t\uparrow\omega.$$

By using this result, relation (3.17), and the fact that the function $\Phi_1^{-1}(z)\,\dfrac{\Phi_1'\left(\Phi_1^{-1}(z)\right)}{z}$ is slowly varying as $z\to Z_1$ (see Remark 2.2), we arrive at the third condition of the theorem in (3.6).

Sufficiency. Assume that conditions (3.1)–(3.6) of the theorem are satisfied.

Applying the transformation

$$\begin{cases}\Phi_1(y'(t))=I_1(t)[1+v_1(x)],\\ \dfrac{y'(t)}{y(t)}=\dfrac{\lambda_0}{\lambda_0-1}\,\dfrac{1}{\pi_\omega(t)}[1+v_2(x)],\end{cases}\tag{3.20}$$

where

$$x=\beta\ln|I_1(t)|,\qquad \beta=\begin{cases}1 & \text{for}\quad \lim\limits_{t\uparrow\omega}I_1(t)=\infty,\\ -1 & \text{for}\quad \lim\limits_{t\uparrow\omega}I_1(t)=0,\end{cases}$$

to Eq. (1.1), we get

$$y'(t)=\Phi_1^{-1}(I_1(t(x))[1+v_1(x)]),$$

$$y(t) = \frac{(\lambda_0 - 1)\Phi_1^{-1}(I_1(t(x))[1 + v_1(x)])\pi_\omega(t)}{\lambda_0[1 + v_2(x)]}.$$

We now reduce system (3.20) to the following system:

$$\begin{cases} 1 + v_1 = \dfrac{\Phi_1^{-1}(I_1(t(x)))}{I_1(t(x)}, \\ 1 + v_2 = \dfrac{(\lambda_0 - 1)y'(t(x))\pi_\omega(t)}{\lambda_0 y(t(x))}. \end{cases} \tag{3.21}$$

This yields

$$v_1' = \beta[1 + v_1]\left[\frac{\Phi_1'(y'(t)I_1(t)y''(t)}{\Phi_1'(y'(t)I_1'(t)} - 1\right]. \tag{3.22}$$

It follows from (1.1) that

$$\begin{aligned} &\frac{\Phi_1'(y'(t))I_1(t)y''(t)}{\Phi_1'(y'(t))I_1'(t)} \\ &\quad= \frac{\Phi_1'(y'(t))I_1(t)}{\Phi_1'(y'(t))I_1'(t)}\alpha_0 p(t)\varphi_1(y)\varphi_0(y') \\ &\quad= \frac{\Phi_1'(y'(t))I_1(t)}{\Phi_1'(y'(t))I_1'(t)}\alpha_0 p(t)\theta_1(y)|y(t)|^{\sigma_1}\varphi_0(y') \\ &\quad= \frac{\alpha_0 p(t)\Phi_1'(y'(t))I_1(t)}{\Phi_1'(y'(t))I_1'(t)}\theta_1\left(\frac{(\lambda_0 - 1)\Phi_1^{-1}(I_1(t(x))[1 + v_1(x)])\pi_\omega(t)}{\lambda_0[1 + v_2(x)]}\right) \\ &\qquad\times\left|\frac{(\lambda_0 - 1)}{\lambda_0}\pi_\omega(t)\right|^{\sigma_1}|y'(t)|^{\sigma_1}[1 + v_2]^{-\sigma_1}\varphi_0(y') \\ &\quad= \frac{\left(\Phi_1'(Y(t, v_1))\right)^2}{\Phi_1''(Y(t, v_1))\Phi_1(Y(t, v_1))}\,\frac{I'(t)I_1(t)}{I_1'(t)I(t)}\,Q(t)\,N(t, v_1, v_2)\,M_1(t, v_1)[1 + v_2]^{-\sigma_1}[1 + v_1]^{-1} \\ &\quad= W(t, v_1, v_2)[1 + v_2]^{-\sigma_1}[1 + v_1]^{-1}, \end{aligned}$$

where

$$W(t, v_1, v_2) = M_1(t(x), v_1)M(t(x), v_1, v_2)N(t, v_1, v_2)\,Q(t)\,\frac{I'(t)I_1(t)}{I_1'(t)I(t)},$$

$$N(t, v_1, v_2) = \frac{\theta_1\left(\dfrac{(\lambda_0 - 1)Y(t, v_1)\pi_\omega(t)}{\lambda_0[1 + v_2]}\right)}{\theta_1\left(|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0 - 1}}\operatorname{sign} y_0^0\right)}, \quad Y(t, v_1) = \Phi_1^{-1}(I_1(t)[1 + v_1]),$$

$$N_1(t, v_1) = \frac{\Phi_1^{-1}(I_1(t))\Phi_1'\left(\Phi_1^{-1}(I_1(t))\right)}{Y(t, v_1)\Phi_1'(Y(t, v_1))}, \quad Q(t) = \frac{\Phi_0\left(\Phi_1^{-1}(I_1(t))\right)}{I(t)},$$

$$M_1(t, v_1) = \frac{\Phi_0\left(\Phi_1^{-1}(I_1(t))\right)[1 + v_1]}{\Phi_0(Y(t, v_1))}, \quad M(t, v_1, v_2) = \frac{\left(\Phi_1'(Y(t, v_1))\right)^2}{\Phi_1''(Y(t, v_1))\Phi_1(Y(t, v_1))}.$$

Thus, relation (3.22) takes the form

$$v_1' = \beta\left[W(t, v_1, v_2)[1 + v_2]^{-\sigma_1} - [1 + v_1]\right].$$

We also get

$$v_2' = \frac{\lambda_0 - 1}{\lambda_0}\,\frac{\beta I_1(t)}{I_1'(t)}\left[\frac{y''(t)\pi_\omega(t)}{y(t)} + \frac{y'(t)\pi_\omega(t)}{y(t)} - \frac{(y'(t))^2\pi_\omega(t)}{y^2(t)}\right]$$

$$= \beta G(t)[1 + v_2]\left[\frac{y''(t)\pi_\omega(t)}{y'(t)} + 1 - \frac{\lambda_0}{\lambda_0 - 1}[1 + v_2]\right],$$

where

$$G(t) = \frac{I_1(t)}{\pi_\omega(t)I_1'(t)}.$$

Note that

$$\frac{y''(t)\pi_\omega(t)}{y'(t)} = \left[\frac{y''(t)\Phi_1'(Y(t, v_1))I_1(t)}{\Phi_1(Y(t, v_1))I_1'(t)}\right]\frac{\Phi_1(Y(t, v_1))}{y'(t)\Phi_1'(Y(t, v_1))}\,\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}$$

$$= W(t, v_1, v_2)[1 + v_2]^{-\sigma_1}[1 + v_1]^{-1}\frac{\pi_\omega(t)I_1'(t)}{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}$$

$$\times \frac{\Phi_1(Y(t, v_1))\Phi_1^{-1}(I_1(t))\Phi_1'\left(\Phi_1^{-1}(I_1(t))\right)}{Y(t, v_1)\Phi_1'(Y(t, v_1))\Phi_1(\Phi_1^{-1}(I_1(t)))}$$

$$= W(t, v_1, v_2)[1 + v_2]^{-\sigma_1}[1 + v_1]^{-1}F(t)V(t, v_1, v_2),$$

where

$$V(t, v_1, v_2) = \frac{\Phi_1^{-1}(I_1(t))\dfrac{\Phi_1'(\Phi_1^{-1}(I_1(t)))}{I_1(t)}}{\Phi_1^{-1}(Y(t, v_1))\,\dfrac{\Phi_1'(\Phi_1^{-1}(Y(t, v_1)))}{Y(t, v_1)}}.$$

Thus, system (3.21) is reduced to the following system:

$$\begin{cases} v_1' = \beta\,[W(t, v_1, v_2)[1+v_2]^{-\sigma_1} - [1+v_1]], \\ v_2' = \beta G(t)[1+v_2]\Bigg[W(t, v_1, v_2)V(t, v_1, v_2)F(t)[1+v_1]^{-1}[1+v_2]^{-\sigma_1} \\ \qquad\qquad\qquad\qquad + 1 - \dfrac{\lambda_0}{\lambda_0 - 1}\,[1+v_2]\Bigg]. \end{cases} \tag{3.23}$$

We consider this system of differential equations on the following set:

$$\Omega = [x_0, +\infty[\times D, \quad \text{where} \quad x_0 = \beta \ln|\pi_\omega(t_0)|,$$

$$D = \left\{(v_1, v_2)\colon |v_i| \le \frac{1}{2},\ i = 1, 2\right\}.$$

In view of the properties of the function Φ_1 (see Remark 2.2), the following relations are true:

$$\lim_{t\uparrow\omega} N_1(t, v_1) = 1 \quad \text{uniformly in} \quad v_1, v_2\colon (v_1, v_2) \in D, \tag{3.24}$$

$$\lim_{t\uparrow\omega} M_1(t, v_1) = 1 \quad \text{uniformly in} \quad v_1, v_2\colon (v_1, v_2) \in D,$$

$$\lim_{t\uparrow\omega} M(t, v_1, v_2) = 1 \quad \text{uniformly in} \quad v_1, v_2\colon (v_1, v_2) \in D,$$

$$\lim_{t\uparrow\omega} V(t, v_1, v_2) = 1 \quad \text{uniformly in} \quad v_1, v_2\colon (v_1, v_2) \in D.$$

By using conditions (3.6), we obtain

$$\lim_{t\uparrow\omega} \frac{I'(t)I_1(t)}{I_1'(t)I(t)} = 1, \qquad \lim_{t\uparrow\omega} F(t) = \frac{1}{\lambda_0 - 1}, \qquad \lim_{t\uparrow\omega} Q(t) = 1.$$

It follows from condition (3.5) that

$$\lim_{t\uparrow\omega} G(t) = 0. \tag{3.25}$$

We prove that

$$\lim_{t\uparrow\omega} N(t, v_1, v_2) = 1 \quad \text{uniformly for} \quad |v_1| < \frac{1}{2},\ \ |v_2| < \frac{1}{2}. \tag{3.26}$$

Note that

$$\lim_{t\uparrow\omega}\frac{\left(\dfrac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}}\right)'\pi_\omega(t)}{\dfrac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}}}=\lim_{t\uparrow\omega}F(t)M(t,v_1)-\frac{1}{(\lambda_0-1)}=0.$$

Hence, the function

$$\left(\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}}\right)$$

is a normalized slowly varying function as $t\uparrow\omega$. Thus, in view of the facts that the function Φ_1^{-1} is slowly varying as the argument tends to Z_1 and the function θ_1 satisfies the condition S, we obtain (3.26).

We now rewrite system (3.23) in the form

$$\begin{cases} v_1' = \beta\,[A_{11}(t)v_1 + A_{12}(t)v_2 + R_1(x,v_1,v_2) + R_2(x,v_1,v_2)], \\ v_2' = \beta G(t)\,[A_{21}v_1 + A_{22}v_2 + R_3(t,v_1,v_2) + R_4(t,v_1,v_2)], \end{cases} \tag{3.27}$$

where

$$A_{11}=1,\quad A_{12}(t)=-\sigma_1,$$

$$R_1(t,v_1,v_2)=(W(t,v_1,v_2)-1)(1-\sigma_1 v_2),$$

$$R_2(t,v_1,v_2)=W(t,v_1,v_2)\left([1+v_2]^{-\sigma_1}-1+\sigma_1 v_2\right),$$

$$A_{21}=-1,\quad A_{22}=\frac{-\lambda_0+\sigma_1}{\lambda_0-1},$$

$$R_3(t,v_1,v_2)=\left(V(t,v_1,v_2)W(t,v_1,v_2)F(t)-\frac{1}{\lambda_0-1}\right)(1-v_1+(1-\sigma_1)v_2),$$

$$\begin{aligned}R_4(t,v_1,v_2)={}&-\frac{\lambda_0}{\lambda_0-1}\,v_2^2+\frac{1}{\lambda_0-1}\,(1+v_1)^{-1}\left([1+v_2]^{1-\sigma_1}-1-v_2(1-\sigma_1)\right)\\ &+\frac{1}{\lambda_0-1}\,\left[(1+v_1)^{-1}-1+v_1\right][1+v_2]^{1-\sigma_1}\\ &+\left(V(t,v_1,v_2)W(t,v_1,v_2)F(t)-\frac{1}{\lambda_0-1}\right)\\ &\times\left([1+v_1]^{-1}[1+v_2]^{1-\sigma_1}-1+v_1-(1-\sigma_1)v_2\right).\end{aligned}$$

In view of (3.24) and (3.25), for $k \in \{2, 4\}$, we get

$$\lim_{|v_1|+|v_2|\to 0} \frac{R_k(t, v_1, v_2)}{|v_1| + |v_2|} = 0 \quad \text{uniformly in} \quad t \in [t_0, \omega[$$

and, for $k \in \{1, 3\}$,

$$\lim_{t\uparrow\omega} R_k(t, v_1, v_2) = 0 \quad \text{uniformly in} \quad v_1, v_2\colon (v_1, v_2) \in D.$$

Since condition (3.2) of the theorem is satisfied, the following finite or infinite limit exists:

$$\lim_{t\uparrow\omega} \frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}}.$$

We now prove that

$$\lim_{t\uparrow\omega} \frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}} = 0.$$

According to the condition (3.2) of the theorem, the following finite or infinite limit exists:

$$\lim_{t\uparrow\omega} \frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}}.$$

Assume the contrary. Let

$$\frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}} = q_1(t) \qquad \text{and} \qquad \lim_{t\uparrow\omega} q_1(t) \neq 0. \tag{3.28}$$

Thus, we get

$$\frac{G'(t)}{\sqrt{|G(t)|}} = \frac{q_1(t)}{\pi_\omega(t)}.$$

Integrating this equality from t_0 to t, we obtain

$$2\sqrt{|G(t)|} - 2\sqrt{|G(t_0)|} = \int_{t_0}^{t} \frac{q_1(\tau)}{\pi_\omega(\tau)}\, d\tau. \tag{3.29}$$

It follows from (3.25) and (3.29) that the integral $\int_{t_0}^{t} \frac{q_1(\tau)}{\pi_\omega(\tau)} d\tau$ must converge but this is possible only in the case where

$$\lim_{t\uparrow\omega} q_1(t) = 0,$$

which contradicts (3.28).

Note that the characteristic equation of the matrix of system (3.27)

$$\begin{pmatrix} -1 & -\sigma_1 \\ \dfrac{-1}{\lambda_0 - 1} & \dfrac{-\lambda_0 - \sigma_1}{\lambda_0 - 1} \end{pmatrix}$$

has the form

$$\mu^2 + \frac{2\lambda_0 - 1 + \sigma_1}{\lambda_0 - 1}\,\mu - \frac{\sigma_1}{\lambda_0 - 1} = 0. \tag{3.30}$$

According to condition (3.1), this equation does not have roots with zero real part. Consider the integral $\int_{x_0}^{\infty} G(t(x))dx$. By using the representation

$$G(t(x)) = \frac{I(t(x))}{\pi_\omega(t(x))I'(t(x))},$$

we obtain

$$\int\limits_{x_0}^{\infty} G(t(x))dx = \int\limits_{x_0}^{\infty} \frac{I_1(t(x))}{\pi_\omega(t(x))I_1'(t(x))}\,dx = \int\limits_{t(x_0)}^{\omega} \frac{I_1(t)}{\pi_\omega(t)I_1'(t)}\,\frac{I_1'(t)}{I_1(t)}\,dt = \ln|\pi_\omega(t)|_{d_1}^{\omega} \to \infty$$

as $t \to \omega$.

In view of the fact that the inequality

$$\int\limits_{x_0}^{\infty} \sqrt{|G(t(x))|}dx \geq \operatorname{sign}\,(G(t(x))) \int\limits_{x_0}^{\infty} G(t(x))dx$$

holds in a neighborhood of zero, we get

$$\int\limits_{x_0}^{\infty} \sqrt{|G(t(x))|}dx = +\infty.$$

Thus, all conditions of Theorem 2.6 in [4] are satisfied for the system of differential equations (3.27). By virtue of this theorem and conditions (3.1), system (3.27) has at least one solution $\{\omega_i\}_{i=1}^{2}\colon [x_1, +\infty[\longrightarrow \mathbb{R}^2$, $x_1 \geq x_0$, approaching zero as $x \to +\infty$. In view of (3.20), this solution corresponds to the solutions y of Eq. (1.1) admitting the asymptotic representations (3.7) as $t \uparrow \omega$.

In view of the form of these representations and (3.1), it is clear that the obtained solution is a $P_\omega(Y_0, Y_1, \lambda_0)$-solution.

The theorem is proved.

Conclusions

In the present paper, for the classes of second-order differential equations (1.1) whose right-hand sides contain the product of a regularly varying nonlinearity of the unknown function and a rapidly varying nonlinearity of the derivative of unknown function as its arguments tend either to zero or to infinity, we establish necessary and sufficient conditions for the existence of regularly varying $P_\omega(Y_0, Y_1, \lambda_0)$-solutions with $\lambda_0 \in \mathbb{R}\backslash\{0, 1\}$. We also constructed the asymptotic representations of these solutions and their first-order derivatives. Note that, under the additional conditions imposed on the coefficients of the characteristic equation (3.30), there exists either a one-parameter or a two-parameter family of these $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of Eq. (1.1).

In [5], similar results were obtained in analyzing equations of the second order whose right-hand sides contain the product of a rapidly varying nonlinearity of the unknown function and a regularly varying nonlinearity of the derivative of the unknown function as its arguments tend either to zero or to infinity.

For Eq. (1.1), the accumulated results are new.

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