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On inverse problem for tree of Stieltjes strings

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Abstract. For a given metric tree and two strictly interlacing sequences of numbers there exits a distribution of point masses on the edges (which are Stieltjes strings) such that one of the sequences is the spectrum of the spectral problem with the Neumann condition at the root of the tree while the second sequence is the spectrum of the spectral problem with the Dirichlet condition at the root.

1. Introduction

Finite-dimensional spectral problems on an interval were considered in [5] (some recent results see in [16, 17] and applications in [1, 4]). Finite-dimensional spectral problems on graphs occur in various fields of physics: mechanics of transverse vibrations of strings [6–8], longitudinal vibrations of point masses joined by springs [11], synthesis of electrical circuits [3, 9]. By inverse problem we mean recovering the parameters of the problem using the spectra of system vibrations. Such a problem on an interval was completely solved in [5], see [18] for generalization. For a star graph the inverse problem was solved in [2, 14, 15, 19]. In these papers conditions on sequences of numbers where obtained necessary and sufficient to be the spectra of spectral problems on the whole graph and on the edges of it. It was shown that the solution of this inverse problem is unique only in the case of simple spectra.

Spectral problems on trees were considered in [12] where it was assumed that the interior vertices of the tree are free of point masses. In this paper we consider the case where point masses can be presented at the interior vertices.

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In Section 2 we describe the spectral problems generated by Stieltjes string equations on a tree with Dirichlet and Neumann conditions at the root which is any of pendant vertices.

Section 3 contains some auxiliary results on Stieltjes functions.

In Section 4 we solve the direct problem, i.e. we show that the quotient of characteristic polynomials of the Dirichlet and Neumann problems can be expanded into branching continued fraction.

In Section 5 we solve the inverse problem of recovering values of point masses and the lengths of subintervals between them using the total lengths of the edges, the spectra of two problems: with the Dirichlet condition at the root, first, and the Neumann condition, the second (Theorem 5.1). Also we consider the inverse problem of recovering values of point masses and the lengths of subintervals between them using the numbers of masses on the edges, the spectra of two problems: with the Dirichlet condition at the root, first, and the Neumann condition, the second (Theorem 5.2).

2. Statement of spectral problems

Let T be a plane metric tree with $q \geqslant 2$ edges. We denote by v_i the vertices, by $d(v_i)$ their degrees, by e_j the edges, by l_j their lengths by $n_j \geqslant 0$ the numbers of point masses $m_1^{(j)}$, $m_2^{(j)}$, …, $m_{n_j}^{(j)}$ which divide the string into the subintervals $l_0^{(j)}$, $l_1^{(j)}$,…, $l_{n_j}^{(j)}$ ($l_k^{(j)} > 0$ for $j = 0, 1, n_j - 1$, $l_{n_j}^{(j)} \geqslant 0$, $m_k^{(j)} > 0$, $l_j = \sum\limits_{k=0}^{n_j} l_k^{(j)}$). The case of $l_{n_j}^{(j)} = 0$ corresponds to location of $m_{n_j}^{(j)}$ at an interior vertex. An arbitrary vertex is chosen to be the root. The pendant vertices are free of masses. All the edges we direct away from the root.

The root $\mathbf{v}$ is the beginning of a subinterval of length $l_0^{(j)}$ on each edge e_j incident with the root. Each other vertex v_i have one incoming edge e_j ending with a subinterval of the length $l_{n_j}^{(j)}$, while each outgoing edge e_r begins at v_i with an interval of lengths $l_0^{(r)}$.

The degree of a vertex v_i is denoted by $d(v_i)$, the indegree of it by $d^+(v_i)$ while the outdegree by $d^-(v_i)$. It is clear that $d^+(v_i) = 1$ for each $v_i \neq \mathbf{v}$ and $d^+(\mathbf{v}) = 0$. At each pendant vertex v_i which is not the root we have $d^-(v_i) = 0$.

It is assumed that the tree is stretched and the pendant vertices except the root are fixed. We consider two cases: in the first the root is fixed (Dirichlet problem) while in the second the root is free to move in the direction orthogonal to the equilibrium position of the tree (Neumann problem). The tree can vibrate in the direction orthogonal to the equilibrium position of the strings. The transverse displacement of the mass $m_k^{(j)}$ we denote by $w_k^{(j)}(t)$, the displacement of the root by $\mathbf{w}(t)$ where t is time.

If an edge e_j is incoming for an interior vertex v_i then the displacement of the incoming end of the edge is denoted by $w_{n_j+1}^{(j)}(t)$, while if an edge e_r is outgoing for a vertex v_i then the displacement of the outgoing end of the edge is denoted by $w_0^{(r)}(t)$. Using such notation vibrations of the tree can be described by the system of equations

$$\frac{w_k^{(j)}(t)-w_{k+1}^{(j)}(t)}{l_k^{(j)}}+\frac{w_k^{(j)}(t)-w_{k-1}^{(j)}(t)}{l_{k-1}^{(j)}}+m_k^{(j)}\frac{\partial^2 w_k^{(j)}}{\partial t^2}(t)=0, \tag{2.1}$$

$$(k=1,2,\ldots,\tilde{n}_j;\quad j=1,2,\ldots,q),$$

where $\tilde{n}_j=n_j-1$ if $l_{n_j}^{(j)}=0$ and $\tilde{n}_j=n_j$ if $l_{n_j}^{(j)}>0$.

For each interior vertex v_i (except of the root) with incoming edge e_j and outgoing edges e_r we impose the continuity conditions

$$w_0^{(r)}(t)=w_{\tilde{n}_j+1}^{(j)}(t) \tag{2.2}$$

for all r corresponding to outgoing edges. Balance of forces at such a vertex v_i implies

$$\frac{w_{\tilde{n}_j+1}^{(j)}(t)-w_{\tilde{n}_j}^{(j)}(t)}{l_{\tilde{n}_j}^{(j)}}+\sum_r\frac{w_0^{(r)}(t)-w_1^{(r)}(t)}{l_0^{(r)}}=\begin{cases}0, & \text{if } l_{n_j}^{(j)}>0,\\ m_{n_j}^{(j)}\dfrac{\partial^2 w_{n_j}^{(j)}(t)}{\partial t^2}, & \text{if } l_{n_j}^{(j)}=0.\end{cases} \tag{2.3}$$

where the sum is taken over all the outgoing edges. For an edge e_j incident with a pendant vertex (except of the root) we impose the Dirichlet boundary condition:

$$w_{n_j+1}^{(j)}(t)=0. \tag{2.4}$$

At the root we will consider the Dirichlet condition

$$\mathbf{w}(t)=0,\quad \text{i.e.}\quad w_0^{(j)}(t)=0 \tag{2.5}$$

for all j corresponding to the edges incident with the root. We call problem (2.1)-(2.5) the Dirichlet boundary value problem on T.

To obtain the Neumann boundary value problem we change (2.5) for the generalized Neumann conditions

$$w_0^{(j)}(t)=w_0^{(l)}(t) \tag{2.6}$$

for all j and l corresponding to the edges incident with the root and

$$\sum_{j=1}^{d(\mathbf{v})}\frac{w_0^{(j)}(t)-w_1^{(j)}(t)}{l_0^{(j)}}=0. \tag{2.7}$$

We exclude from consideration the case with a mass at the root.

If the root is a pendant vertex then we denote by e_1 the edge incident with the root. In this case the Dirichlet condition (2.5) at the root is

$$\mathbf{w}(t)=0, \quad \text{i.e.} \quad w_0^{(1)}(t)=0 \tag{2.8}$$

and the Neumann conditions (2.6), (2.7) in this case can be reduced to

$$\mathbf{w}(t)=w_0^{(1)}(t)=w_1^{(1)}(t). \tag{2.9}$$

Substituting $w_j^{(k)}(t)=e^{i\lambda t}u_j^{(k)}$, $w_i(t)=e^{i\lambda t}u_i$ and $\mathbf{w}(t)=e^{i\lambda t}\mathbf{u}$ into (2.1)-(2.9) we obtain the corresponding spectral problems:

Dirichlet problem. For each edge:

$$\frac{u_k^{(j)}-u_{k+1}^{(j)}}{l_k^{(j)}}+\frac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}-m_k^{(j)}\lambda^2u_k^{(j)}=0, \tag{2.10}$$

$$(k=1,2,\ldots,\tilde{n}_j,\ j=1,2,\ldots,q).$$

For each interior vertex (except for the root) with incoming edge e_j and outgoing edges e_r we have

$$u_{\tilde{n}_j+1}^{(j)}=u_0^{(r)}, \tag{2.11}$$

and

$$\frac{u_{\tilde{n}_j+1}^{(j)}-u_{\tilde{n}_j}^{(j)}}{l_{\tilde{n}_j}^{(j)}}+\sum_r\frac{u_0^{(r)}-u_1^{(r)}}{l_0^{(r)}}=\begin{cases}0, & \text{if } l_{n_j}^{(j)}>0,\\ -m_{n_j}^{(j)}\lambda^2u_{n_j}^{(j)}, & \text{if } l_{n_j}^{(j)}=0.\end{cases} \tag{2.12}$$

For an edge e_j incident with a pendant vertex (except of the root) we have the Dirichlet boundary condition:

$$u_{n_j+1}^{(j)}=0. \tag{2.13}$$

At the root we have the Dirichlet condition

$$\mathbf{u}=0, \quad \text{i.e.} \quad u_0^{(j)}=0 \tag{2.14}$$

for all j corresponding to the edges incident with the root.

If the root is a pendant vertex and e_1 the edge incident with the root then instead of (2.14) at the root we have

$$u_0^{(1)}=0. \tag{2.15}$$

Neumann problem. The Neumann problem on T consists of equations (2.10)-(2.13), of

$$u_0^{(j)} = u_0^{(l)} \tag{2.16}$$

for all j and l corresponding to the edges incident with the root and of

$$\sum_{j=1}^{d(\mathbf{v})} \frac{u_0^{(j)} - u_1^{(j)}}{l_0^{(j)}} = 0. \tag{2.17}$$

If the root is a pendant vertex and e_1 the edge incident with the root then instead of (2.16) and (2.17) at the root we have

$$u_0^{(1)} = u_1^{(1)}. \tag{2.18}$$

3. Auxiliary results

Here we give some lemmas which are used in Section 4.

Definition 3.1. A rational function $f(z)$ is said to be a *Nevanlinna function* if:

(i) it is analytic in the half-planes $\text{Im}z > 0$ and $\text{Im}z < 0$;
(ii) $f(\overline{z}) = \overline{f(z)}$ $(\text{Im}z \neq 0)$;
(iii) $\text{Im}z\ \text{Im}f(z) \geqslant 0$ for $\text{Im}z \neq 0$.

Definition 3.2. A rational Nevanlinna function $f(z)$ is said to be an *S-function* if $f(z) > 0$ for $z < 0$.

Definition 3.3. A rational S-function $f(z)$ is said to be an S_0*-function* if 0 is not a pole of $f(z)$.

The following lemmas are obvious.

Lemma 3.4. *Suppose that f and g are rational S_0-functions, then $f + g$ and $(f^{-1} + g^{-1})^{-1}$ are also S_0-functions.*

Lemma 3.5. *Let ϕ_j be a rational S_0 function with n_j zeros and n_j poles for $j = 1, 2, \ldots, s$. Then*

$$\frac{1}{\sum_{j=1}^{s} \frac{1}{\phi_j}}$$

is a rational S_0 function with $\tilde{n}$ zeros and $\tilde{n}$ poles where $\tilde{n} \geqslant max\{n_1, n_2, \ldots, n_s\}$.

Lemma 3.6. (see [14, Lemma 2.2]) *Let*

$$\varphi = a_0 + \cfrac{1}{-b_1 z + \cfrac{1}{a_1 + \cfrac{1}{-b_2 z + \ldots + \cfrac{1}{a_{r-1} + \cfrac{1}{-b_r z + \cfrac{1}{a_r + \phi}}}}}}. \tag{3.1}$$

where $a_j > 0$ for $j = 0, 1, \ldots, r-1$, $a_r \geqslant 0$, $b_j > 0$ for $j = 1, 2, \ldots, r$ and let ϕ be a rational S_0-function with $\hat{n}$ poles and $\hat{n}$ zeros. Then φ is a rational S_0-function with $\hat{n}+r$ zeros and $\hat{n}+r$ poles.

Lemma 3.7. *A rational function*

$$f(z) = C \prod_{k=1}^{n} \frac{1 - \dfrac{z}{(\nu_k)^2}}{1 - \dfrac{z}{(\mu_k)^2}}, \tag{3.2}$$

is an S_0-function if and only if $C > 0$ and

$$0 < (\mu_1)^2 < (\nu_1)^2 < \ldots < (\mu_n)^2 < (\nu_n)^2.$$

Lemma 3.8. (see e.g. [10, Chapter II.2, p.19/26]). *A rational function*

$$f(z) = C \prod_{k=1}^{n} \frac{1 - \dfrac{z}{(\nu_k)^2}}{1 - \dfrac{z}{(\mu_k)^2}}, \tag{3.3}$$

is an S_0-function if and only if $0 < (\nu_1)^2 < (\nu_2)^2 < \ldots . < (\nu_k)^2$ and

$$f(z)^{-1} = \sum_{k=1}^{n} \frac{A_k}{z - (\nu_k)^2} + B$$

where $A_k > 0$ for $k = 1, 2, \ldots, n$ and

$$B > \sum_{k=1}^{n} \frac{A_k}{(\nu_k)^2}.$$

4. Direct problem

Here and in the sequel we consider a tree T rooted at a pendant vertex. The Dirichlet problem on this tree consists of equations (2.10)-(2.13) and (2.15), while the Neumann problem on it consists of (2.10)-(2.13) and (2.18).

First of all we notice that interior vertices of degree 2 do not influence the results and we can assume absence of such vertices without losses of generality. Let P be a path in the tree T involving the maximum number of masses. Obviously it starts and finishes with pendant vertices. We denote the initial vertex by v_0 and choose it as the root of the tree. The enumeration of other vertices is arbitrary. We direct the edges away from the root. Denote by e_i the edge incoming into a vertex v_i for all i. Then

$$P : v_0 \to v_1 \to v_{s_2} \to v_{s_3} \to \ldots \to v_{s_{r-1}} \to v_{s_r}.$$

Here r is a combinatorial length of the path. Deleting v_0 and e_1 we obtain a new tree T' rooted at the vertex v_1.

Since $d(v_1) > 2$ we can divide our tree T' into subtrees

$$T'_1,\ T'_2, \ldots,\ T'_{d(v_1)-1}$$

having v_1 as the only common vertex. (We say that T'_1, T'_2,..., $T'_{d(v_1)-1}$ are *complementary subtrees* of T' (see Fig. 4.1).

Denote by $\phi_N(\mathbf{v})$ the characteristic polynomial of problem (2.10)-(2.13), (2.18) on the tree T and by $\phi_D(\mathbf{v})$ the characteristic polynomial of problem (2.10)-(2.13), (2.15) on this tree. These polynomials are normalized such that

$$\frac{\phi_{D(v)}(0)}{\phi_{N(v)}(0)} = l_1 + \frac{1}{\sum\limits_{r=1}^{d(v_1)-1} \frac{\phi_{N_r(v_1)}(0)}{\phi_{D_r(v_1)}(0)}},$$

$\phi_{D,r(v_1)}(z)$ is the characteristic polynomial of the Dirichlet problem (2.10)-(2.13), (2.15) on T'_r and $\phi_{N,r(v_1)}(z)$ is the characteristic polynomial of the Neumann (2.10)-(2.13), (2.18) on T'_r and this expansion can be continued.

The following result was proved in [12] (see the proof of Corollary 2.9 there). In that paper it was assumed absence of point masses at the interior vertices of the tree ($l_{n_j}^{(j)} > 0$ for all j) but the proof remains true if $l_{n_j}^{(j)} = 0$ for some values of j.

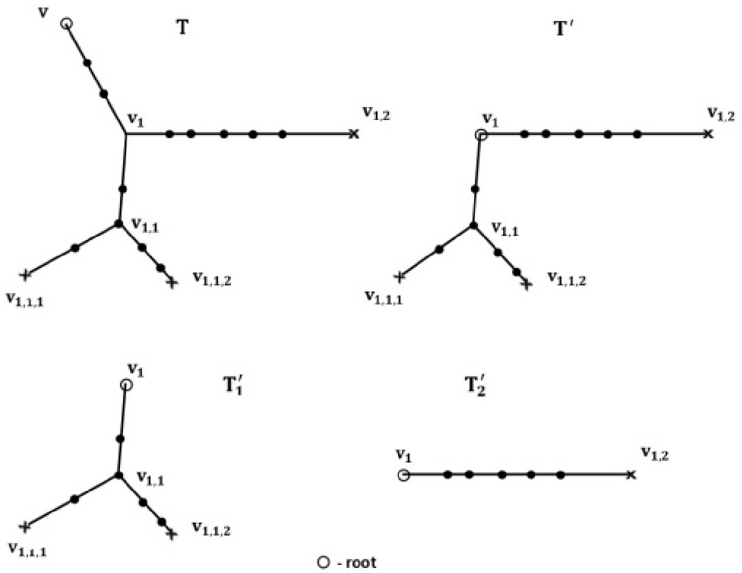

FIGURE 4.1.

Theorem 4.1. *Let the root* $\mathbf{v}$ *be a pendant vertex of a tree* T. *Then the quotient* $\frac{\phi_{D(\mathbf{v})}(z)}{\phi_{N(\mathbf{v})}(z)}$ *can be expanded in continued fraction*

$$\frac{\phi_{D(\mathbf{v})}(z)}{\phi_{N(\mathbf{v})}(z)} = l_0^{(1)} + \cfrac{1}{-m_1^{(1)}z + \cfrac{1}{l_1^{(1)} + \cfrac{1}{-m_2^{(1)}z + \ldots + \cfrac{1}{-m_{n_1}^{(1)}z + \cfrac{1}{l_{n_1}^{(1)} + \frac{\phi_{D(v_1)}(z)}{\phi_{N(v_1)}(z)}}}}}} \tag{4.1}$$

where $\phi_{D(v_1)}(z)$ *is the characteristic polynomial of problem* (2.10)-(2.14) *on* T' *and* $\phi_{N(v_1)}(z)$ *is the characteristic polynomial of problem* (2.10)-(2.13), (2.16), (2.17) *on* T',

$$\frac{\phi_{D(v_1)}(z)}{\phi_{N(v_1)}(z)} = \frac{1}{\sum_{r=1}^{d(v_1)-1} \frac{\phi_{N_r(v_1)}(z)}{\phi_{D_r(v_1)}(z)}} \tag{4.2}$$

where $d(v_1)$ *is the degree of* v_1 *as a vertex of* T, $\phi_{D_r(v_1)}$ *is the characteristic polynomial of problem* (2.10)-(2.13), (2.15) *on* T_r *and* $\phi_{N_r(v_1)}$ *is the* *characteristic polynomial of problem* (2.10)-(2.13), (2.18) *on it.*

$$\frac{\phi_{D_r(v_1)}(z)}{\phi_{N_r(v_1)}(z)} = l_0^{(r)} + \cfrac{1}{-m_1^{(r)}z + \cfrac{1}{l_1^{(r)} + \cfrac{1}{-m_2^{(r)}z + \ldots + \cfrac{1}{-m_{n_r}^{(r)}z + \cfrac{1}{l_{n_r}^{(r)} + f_r(z)}}}}}. \tag{4.3}$$

In turn each f_r *can be expanded similarly to* (4.2), (4.3).

Remark 4.2. It is proved in [13] that the number of distinct eigenvalues of each of the problems (2.10)-(2.13), (2.15) and (2.10)-(2.13), (2.18) on a tree bearing masses on every edge is not less than the maximum number of point masses on a path in this tree.

5. Inverse problem

Theorem 5.1. *Suppose* $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ *and* $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$ *are symmetric* ($\mu_{-k} = -\mu_k$, $\nu_{-k} = -\nu_k$) *and monotonic sequences of real numbers which interlace:*

$$0 < (\mu_1)^2 < (\nu_1)^2 < \ldots < (\mu_n)^2 < (\nu_n)^2. \tag{5.1}$$

Let T *be a metric tree of a prescribed formrooted at a pendant vertex* $\mathbf{v}$ *with prescribed lengths of edges* $l_j > 0$ ($j = 1, 2, \ldots, q$, q *is the number of edges in* T). *Then*

1) *there exist numbers* $n_j \in \{0\} \cup \mathbb{N}$ ($j = 1, 2, \ldots, q$), *sequences of positive numbers* $\{m_k^{(j)}\}_{k=1}^{n_j}$ (*point masses on the edge* e_j, $j = 1, 2, \ldots, q$) *and numbers* $\{l_k^{(j)}\}_{k=0}^{n_j}$ ($l_k^{(j)} > 0$ *for all* $k = 0, 1, \ldots, n_j - 1$, $l_{n_j} \geqslant 0$ *for all* $j = 1, 2, \ldots, g$ *such that* $\sum\limits_{k=0}^{n_j} l_k^{(j)} = l_j$, $\sum\limits_{j=1}^{q} n_j = n$, *the spectrum of Neumann problem* (2.10)-(2.13), (2.18), *coincides with* $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ *and the spectrum of Dirichlet problem* (2.10)-(2.13), (2.15) *coincides with* $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$;
2) *the two spectra* $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ *and* $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$ *and the length* l_1 *of the edge incident with the root uniquely determine the masses* $\{m_k^{(1)}\}_{k=1}^{n_1}$ (*point masses on the edge* e_1) *and lengths* $\{l_k^{(1)}\}_{k=0}^{n_1}$ *of the subintervals on this edge.*

Proof. First of all we consider the rational function

$$F(z) := \Phi_{T,\mathbf{v}} \prod_{k=1}^{n} \frac{1 - \dfrac{z}{(\mu_k)^2}}{1 - \dfrac{z}{(\nu_k)^2}}, \tag{5.2}$$

where $\Phi_{T,\mathbf{v}} > 0$ is the form characteristic of the tree T which depends only on the form of the tree and the lengths of the edges. It can be found substituting $z = 0$ in (4.1)-(4.3): $\Phi_{T,\mathbf{v}} := \dfrac{\phi_{N(\mathbf{v})}(0)}{\phi_{D(\mathbf{v})}(0)}$.

Let e_1 be the edge connecting $\mathbf{v}$ with v_1 and let l_1 be the length of this edge. Substituting $z = 0$ in (4.1) we obtain

$$\Phi_{T,\mathbf{v}}^{-1} = l_1 + \frac{\phi_{D(v_1)}(0)}{\phi_{N(v_1)}(0)} > l_1 \tag{5.3}$$

Due to (5.1) and $\Phi_{T,\mathbf{v}} > 0$, $F^{-1}(z)$ is an S_0-function and, therefore can be presented as

$$F(z)^{-1} = a_0 + \cfrac{1}{-b_1 z + \cfrac{1}{a_1 + \cfrac{1}{-b_2 z + \ldots + \cfrac{1}{a_{n-1} + \cfrac{1}{-b_n z + \cfrac{1}{a_n}}}}}}, \tag{5.4}$$

where $a_k > 0$ for $k = 0, 1, \ldots, n$ and $b_k > 0$ for $k = 1, 2, \ldots, n$.

Since $F^{-1}(0) = \sum\limits_{k=0}^{n} a_k = \Phi_{T,\mathbf{v}}^{-1} \geqslant l_1$, we can choose the integer number n_1 such that

$$\sum_{k=0}^{n_1-1} a_k \leqslant l_1 < \sum_{k=0}^{n_1} a_k \tag{5.5}$$

and present $F(z)^{-1}$ as follows:

$$F(z)^{-1} = a_0 + \cfrac{1}{-b_1 z + \cfrac{1}{a_1 + \cfrac{1}{-b_2 z + \ldots + \cfrac{1}{a_{n_1-1} + \cfrac{1}{-b_{n_1} z + \cfrac{1}{\hat{a}_{n_1} + F_1(z)^{-1}}}}}}} \tag{5.6}$$

where

$$\hat{a}_{n_1} = l_1 - \sum_{k=0}^{n_1-1} a_k \tag{5.7}$$

and

$$F_1(z)^{-1} = a_{n_1} - \hat{a}_{n_1} + \cfrac{1}{-b_{n_1+1}z + \cfrac{1}{a_{n_1+1} + \cfrac{1}{-b_{n_1+2}z + \ldots + \cfrac{1}{a_{n-1} + \cfrac{1}{-b_n z + \cfrac{1}{a_n}}}}}}. \tag{5.8}$$

We identify $\{a_0, a_1, \ldots, a_{n_1-1}, \hat{a}_{n_1}\}$ with the subintervals of the edge e_1 and $\{b_1, b_2, \ldots, b_{n_1}\}$ with the masses on it:

$$\begin{aligned} a_k &= l_k^{(1)}, (k = 0, 1, \ldots, n_1 - 1), \\ \hat{a}_{n_1} &= l_{n_1}^{(1)}, \\ b_k &= m_k^{(1)}, (k = 1, 2, \ldots, n_1). \end{aligned}$$

Since $d(v_1) > 2$ in T, we divide the tree T' into $d(v_1) - 1$ complementary subtrees T_j rooted at v_1 each (see Fig. 5.1). It is clear that $F_1(z)^{-1}$ belongs to S_0 and therefore

$$F_1(z) = \Phi_{T',v_1} \prod_{k=1}^{n-n_1} \frac{1 - \dfrac{z}{(\tilde{\mu}_k)^2}}{1 - \dfrac{z}{(\tilde{\nu}_k)^2}}, \tag{5.9}$$

where

$$\Phi^{-1}_{T',v_1} = \Phi^{-1}_{T,\mathbf{v}} - l_1,$$

$$0 < (\tilde{\mu}_1)^2 < (\tilde{\nu}_1)^2 < (\tilde{\mu}_2)^2 < \ldots < (\tilde{\nu}_{n-n_1})^2$$

It is known that if $F_1(z)^{-1}$ belongs to S_0 then

$$F_1(z) = \sum_{k=1}^{n-n_1} \frac{A_k}{z - (\tilde{\nu}_k)^2} + B, \tag{5.10}$$

$$A_k > 0, B > 0 \tag{5.11}$$

and since $F^{-1}(0) = \Phi^{-1}_{T',v_1}$ we have

$$B = \Phi_{T',v_1} + \sum_{k=1}^{n-n_1} \frac{A_k}{(\tilde{\nu}_k)^2}.$$

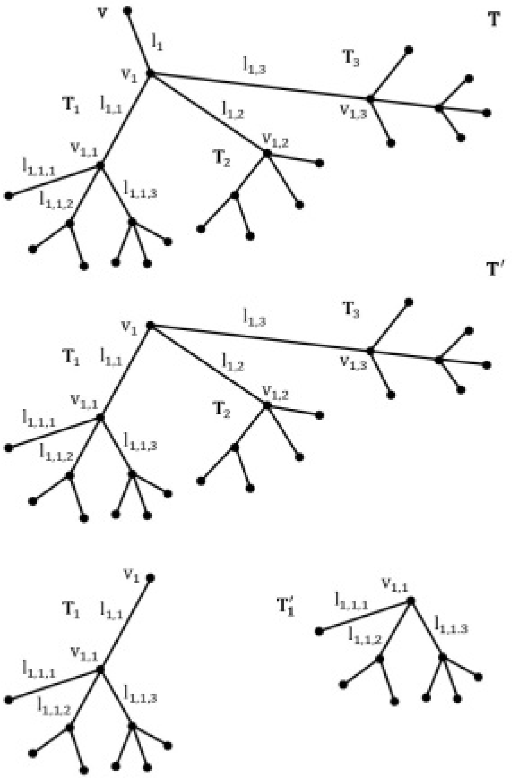

FIGURE 5.1.

Let us choose nonnegative integers N_j $(j = 1, 2, \dots, d(v_1) - 1)$ such that

$$\sum_{j=1}^{d(v_1)-1} N_j = n - n_1.$$

Since $\Phi_{T',v_1} > 0$ and therefore,

$$B > \sum_{k=1}^{n-n_1} \frac{A_k}{(\tilde{\nu}_k)^2} \tag{5.12}$$

We arrange the set $\{(\tilde{\nu}_k)^2\}_{k=1}^{n-n_1}$ as the union of disjoint sets

$$\{(\tilde{\nu}_{k_s^{(1)}})^2\}_{s=1}^{N_1}, \quad \{(\tilde{\nu}_{k_s^{(2)}})^2\}_{s=1}^{N_2}, \quad \dots, \quad \{(\tilde{\nu}_{k_s^{(d(\mathrm{v})-1)}})^2\}_{s=1}^{N_{d(v_1)-1}}$$

and choose numbers B_j
$(j = 1, 2, \ldots, d(v_1) - 1)$ such that

$$\sum_{j=1}^{d(v_1)-1} B_j = B \tag{5.13}$$

and

$$B_j > \sum_{s=1}^{N_j} \frac{A_{k_s}}{(\tilde{\nu}_{k_s}^{(j)})^2}. \tag{5.14}$$

Then

$$F_1(z) = \Phi_{T',v_1} \prod_{k=1}^{n-n_1} \frac{1 - \dfrac{z}{(\tilde{\mu}_k)^2}}{1 - \dfrac{z}{(\tilde{\nu}_k)^2}} = \sum_{j=1}^{d(v_1)-1} \left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z - (\tilde{\nu}_{k_s}^{(j)})^2}{}^{k_s} + B_j \right). \tag{5.15}$$

Put

$$\Phi_{T_j,v_1} = B_j - \sum_{s=1}^{N_j} \frac{A_{k_s}}{(\tilde{\nu}_{k_s}^{(j)})^2} > 0. \tag{5.16}$$

We will show that there exists a distribution of masses on the complimentary subtrees T_j $(j = 1, 2, \ldots, d(v_1) - 1)$, $\bigcup\limits_{j=1}^{d(v_1)-1} T_j = T'$ such that

1) the number of masses in T_j is N_j;
2) the form characteristic of T_j is Φ_{T_j,v_1} and the rational function

$$\sum_{s=1}^{N_j} \frac{A_{k_s}}{z - (\tilde{\nu}_{k_s}^{(j)})^2} + B_j \tag{5.17}$$

having N_j simple zeros and N_j simple poles is $\dfrac{\phi_N^{(j)}(z)}{\phi_D^{(j)}(z)}$, where $\phi_N^{(j)}(z)$ and $\phi_D^{(j)}(z)$ are the characteristic polynomials for the Neumann and the Dirichlet problems on the subtree T_j.

Let us expand $\left(\sum\limits_{s=1}^{N_j}\dfrac{A_{k_s}}{z-(\tilde{\nu}_{k_s}^{(j)})^2}+B_j\right)^{-1}$ into continued fractions

$$\left(\sum_{s=1}^{N_j}\frac{A_{k_s}}{z-(\tilde{\nu}_{k_s}^{(j)})^2}+B_j\right)^{-1}=$$
$$=a_0^{(j)}+\cfrac{1}{-b_1^{(j)}z+\cfrac{1}{a_1^{(j)}+\cfrac{1}{-b_2^{(j)}z+\ldots+\cfrac{1}{a_{N_j-1}^{(j)}+\cfrac{1}{-b_{N_j}^{(j)}z+\cfrac{1}{a_{N_j}^{(j)}}}}}}}, \tag{5.18}$$

It is clear that due to (5.11) and (5.14) the left hand side of (5.18) is an S_0-function.

We define $\tilde{n}_j$ by the inequalities $\sum\limits_{k=0}^{\tilde{n}_j-1}a_k^{(j)}\leqslant l_{1,j}<\sum\limits_{k=0}^{\tilde{n}_j}a_k^{(j)}$ where $l_{1,j}$ is the length of the edge of T_j incident with v_1. Then we can rewrite (5.18) as

$$\left(\sum_{s=1}^{N_j}\frac{A_{k_s}}{z-(\tilde{\nu}_{k_s}^{(j)})^2}+B_j\right)^{-1}=$$
$$=a_0^{(j)}+\cfrac{1}{-b_1^{(j)}z+\cfrac{1}{a_1^{(j)}+\cfrac{1}{-b_2^{(j)}z+\ldots+\cfrac{1}{a_{\tilde{n}_j-1}^{(j)}+\cfrac{1}{-b_{\tilde{n}_j}^{(j)}z+\cfrac{1}{\tilde{a}_{\tilde{n}_j}^{(j)}+F_{j,1}^{-1}(z)}}}}}}, \tag{5.19}$$

where

$$\tilde{a}_{\tilde{n}_j}^{(j)}=l_{1,j}-\sum_{k=0}^{\tilde{n}_j-1}a_k^{(j)}>0$$

and

$$F_{j,1}^{-1}(z) = a_{\tilde{n}_j}^{(j)} - \tilde{a}_{\tilde{n}_j}^{(j)} + \cfrac{1}{-b_{\tilde{n}_j+1}^{(j)}z + \cfrac{1}{a_{\tilde{n}_j+1}^{(j)} + \cfrac{1}{-b_{\tilde{n}_j+2}^{(j)}z + \ldots + \cfrac{1}{a_{N_j-1}^{(j)} + \cfrac{1}{-b_{N_j}^{(j)}z + \cfrac{1}{a_{N_j}^{(j)}}}}}}}. \tag{5.20}$$

We identify numbers $a_0^{(j)}, a_1^{(j)}, \ldots, a_{\tilde{n}_j-1}^{(j)}, \tilde{a}_{\tilde{n}_j}^{(j)}$ with the lengths of subintervals $l_0^{(1,j)}, l_1^{(1,j)}, \ldots, l_{\tilde{n}_j}^{(1,j)}$ and $b_1^{(j)}, b_2^{(j)}, \ldots, b_{\tilde{n}_j}^{(j)}$ with the values of masses $m_1^{(1,j)}, \ldots, m_{\tilde{n}_j}^{(1,j)}$ on the edge $e_{1,j}$. In the same way as (5.9) we obtain

$$F_{j,1}(z) = \sum_{k=1}^{N_j-\tilde{n}_j} \frac{\tilde{A}_k}{z-(\tilde{\nu}_k)^2} + \tilde{B}_j \tag{5.21}$$

with $\tilde{A}_k^{(j)} > 0$ and $\tilde{B}_j > \sum_{k=1}^{N_j-\tilde{n}_j} \frac{\tilde{A}_k}{(\tilde{\nu}_k^{(j)})^2}$.

Denote by $v_{1,j}$ the second vertex incident with $e_{1,j}$ and by $d(v_{1,j})$ the degree of $v_{1,j}$ Now we consider the tree T_j' obtained by deleting the edge $e_{1,j}$ from T_j (see Fig. 5.1). Let $v_{1,j}$ be the root of T_j'.

Substituting $z = 0$ into (5.15) and making use of (5.16) we obtain

$$\Phi_{T',v_1} = \sum_{i=1}^{d(v_1)-1} \Phi_{T_j,v_1}.$$

On the other hand, (5.19) implies

$$\Phi_{T_j,v_1}^{-1} = l_{1,j} + F_{j,1}(0)^{-1}$$

and consequently, $F_{j,1}^{-1}(0) = \Phi_{T_j',v_1}^{-1}$.

We continue this procedure. Finally we obtain a branching continued fraction. We identify

$$\begin{aligned}
a_k^{(j)} &= l_k^{(1,j)}, (k = 0, 1, \ldots, n_j - 1), \\
\hat{a}_{n_j} &= l_{n_j}^{(1,j)}, \\
b_k^{(j)} &= m_k^{(1,j)}, (k = 1, 2, \ldots, n_{1,j}).
\end{aligned}$$

According to Theorem 4.1 the problems (2.10)-(2.13), (2.18) and (2.10)-(2.13), (2.15) with these masses and subintervals have the quotient

$$\frac{\phi_{D(\mathbf{v})}(z)}{\phi_{N(\mathbf{v})}(z)} = F^{-1}(z),$$

where $F(z)$ is given by (5.2).

Since the equation (5.6) uniquely determines the sets $\{a_k\}_{k=0}^{n_1-1} \cup \{\hat{a}_{n_1}\}$ and $\{b_k\}_{k=1}^{n_1}$ and (5.7) uniquely determines the number n_1 Statement 2) is valid. □

Theorem 5.2. *Suppose $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ and $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$ are symmetric ($\mu_{-k} = -\mu_k$, $\nu_{-k} = -\nu_k$) and monotonic sequences of real numbers which interlace:*

$$0 < (\mu_1)^2 < (\nu_1)^2 < \ldots < (\mu_n)^2 < (\nu_n)^2. \tag{5.22}$$

Let T be a metric tree of a prescribed form rooted at a pendant vertex $\mathbf{v}$ with prescribed number of masses on the edges $n_j \geqslant 0$ ($j = 1, 2, \ldots, q$, q is the number of edges in T, $n_j \geqslant 0$, $\sum\limits_{j=1}^{q} n_j = n$).

Then there exist sequences of positive numbers $\{m_k^{(j)}\}_{k=1}^{n_j}$ (point masses on the edge e_j, $j = 1, 2, \ldots, q$) and numbers $\{l_k^{(j)}\}_{k=0}^{n_j}$ ($l_k^{(j)} > 0$ for all $k = 0, 1, \ldots, n_j - 1$, $l_{n_j} \geqslant 0$ for all $j = 1, 2, \ldots, g$ such that the numbers of point masses on the edge e_j is n_j ($j = 1, 2, \ldots, q$) and the spectrum of Neumann problem (2.10)-(2.13), (2.18), coincides with $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ and the spectrum of Dirichlet problem (2.10)-(2.13), (2.15) coincides with $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$.

Proof. We choose an arbitrary positive number $\Phi_{T,\mathbf{v}}$ and construct the rational function $F(z)$ as in (5.2) and expand $F^{-1}(z)$ into continued fraction (5.4), choose l_1 such that satisfies (5.5) and define

$$\Phi_{T',v_1}^{-1} := \Phi_{T,\mathbf{v}}^{-1} - l_1.$$

Since

$$l_1 < \sum_{k=0}^{n_1} a_k \leqslant \sum_{k=0}^{n} a_k = F^{-1}(0) = \Phi_{T,\mathbf{v}}^{-1}$$

we arrive at

$$\Phi_{T',v_1}^{-1} > 0.$$

Now N_j ($j = 1, 2, \ldots, d(v_1)-1$) which is the total number of point masses on T_j can be calculated using the given values of numbers of point masses n_j corresponding to the edges of T_j.

We arrange the set $\{A_k\}_{k=1}^{n-n_1}$ of coefficients in (5.10) into groups $\{A_{k_s}^{(j)}\}_{s=1}^{N_j}$ $(j = 1, 2, \ldots, d(\mathbf{v}) - 1)$ in arbitrary way and choose positive numbers B_j in so that (5.13), (5.14) hold. The latter is possible due to (5.12). Denote by Φ_{T_j,v_1} the values obtained via (5.16) and regarded as the form factors of the trees T_j we are constructing. Due to $A_j > 0$ and (5.14) the functions $\left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z-(\nu_{k_s}^{(j)})^2} + B_j\right)^{-1}$ are S_0-functions and, consequently, (5.19) holds. Let us choose the length l_j of the edge e_j incident with $\mathbf{v}$ so that

$$\sum_{k=0}^{n_j-1} a_k^{(j)} \leqslant l_j < \sum_{k=0}^{n_j} a_k^{(j)},$$

where n_j is the number of the masses on e_j. Then we present the functions as in (5.19) and (5.20) but with given n_j instead of $\tilde{n}_j$. Then we continue this procedure as in the proof of Theorem 5.1. □

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